Ihara zeta function

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Riemann zeta function

Recall that the Riemann zeta function $\zeta(s)$ satisfies

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},\ Re(s) > 1,$$

and there is an analytic continuation.

We also have

$$\zeta(s) = \prod_{p=\text{prime}} (1 - p^{-s})^{-1}$$

(Exercise)

The functional equation is

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(1-s).$$

The Riemann hypothesis is that the nonreal zeros z of $\zeta(s)$ satisfy $Re(z) = \frac{1}{2}$.

The prime number theorem is that

$$\pi(x) = \#\left\{p = \text{prime} : p \leq x\right\} \sim \frac{x}{\log x} \text{ as } x \to \infty$$

The Riemann hypothesis is equivalent to

$$\left|\pi(x) - \int_0^x \frac{dt}{\log(t)}\right| < \frac{1}{8\pi}\sqrt{x}\log(x),$$

for all $x \geq 2657$. (Joke)

Note that n^2 is an eigenvalue of $-\frac{d^2}{d\theta^2}$ on functions on $S^1 = \mathbb{R}/\mathbb{Z}$.

Selberg zeta function

The Riemann uniformization theorem says that any compact surface with $g \geq 2$ admits a complex structure and has the structure $\mathbb{H}/\Gamma$, where Γ is a discrete subgroup of $SL(2,\mathbb{Z})$. It admits a riemannian metric of constant Gauss curvature -1.

A geodesic is a distance-minimizing curve on the manifold. Prime numbers on the manifold are lengths of closed geodesics. The corresponding zeta function is called the Selberg zeta function, which is

$$Z(s) = \prod_{[c]}\prod_{j\geq 1}(1 - \exp(-(s+j)\nu(c))),$$

where $\nu(c)$ is the length of the unique closed geodesic in the free homotopy class $[c]$ of simple closed curves. This is related to the spectrum of the hyperbolic Laplacian

$$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

Graphs and the Ihara zeta function

Primes in a graph are simple closed geodesics, i.e. closed paths that minimize length. Length is simply the number of edges. Let X be a finite, connected, unorientedgraph — ie a collection of vertices and edges. We assume that X is not a cycle or a cycle with hair. Our graphs is permitted to have multiple edges and loops. Let E be the set of edges, and let V be the set of vertices. Orient the edges arbitrarily. We want to distinguish edges with the opposite orientation. Let $e_1,\ldots,e_n,e_{n+1}=e_1^{-1},\ldots,e_{2n}=e_n^{-1}$.

Prime numbers in X is an equivalence class $[c]$ of tailless primitive closed paths in X. A **path** $c=a_1a_2\ldots a_k$ is a list of contiguous edges (head of previous edge must be the tail of the current edge). A **backtrack** $c=a_1a_2\ldots a_k$ is a path such that $a_{j+1}=a_j^{-1}$ for some j. It is a **tail** if $a_k=a_1^{-1}$. Note that $length(c)=k$. A closed path is **prime** (or **primitive**) if it has no backtrack or tail and it is not a power of any other path. The equivalence class of a path is the set of paths where you start at different vertices. Note that the product of different primes may be prime. So there is no unique factorization. The only nonprime paths are powers of primes. Graphs with more than one cycle have an infinite number of primes.

The Ihara zeta function $\zeta(u,X)$ is

$$\zeta(u,X)=\prod_{[p]}(1-u^{\nu(p)})^{-1},$$

where u is a complex variable and the product is over primes, and $\nu(p)$ is the length of the prime p, the number of edges. One can show that there are no zeros but has a punch of poles.

$$\zeta(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2),$$

where A is the **adjacency matrix of** X, Q is a diagonal matrix, where you put the degree of the corresponding vertex minus 1 in each diagonal entry. The number r is the number of loops after homotoping vertices. In fact, the number r is the number of edges minus the number of vertices plus 1. For a graph that is a square with one diagonal, we have

$$\zeta(u,X)^{-1}=(1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3).$$

Lecture Notes #2

Recall definition of Ihara zeta function above.

Theorem $\zeta(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2)$, where A is the **adjacency matrix of** X, Q is a diagonal matrix, where you put the degree of the corresponding vertex minus 1 in each diagonal entry. The number r is the number of loops after homotoping vertices. In fact, the number r is the number of edges minus the number of vertices plus 1.

Example:

Let X be the graph with vertices $1,2,3,4$ and edges $e_1=12,e_2=23,e_3=34,e_4=41,e_5=42$. Then $r=2$. Also, let $e_{j+5}=e_j^{-1}$. A backtrack looks like $e_3e_8=e_3e_3^{-1}$. The closed path like $e_1e_2e_3e_5e_1^{-1}$ is one with a tail. Some examples of a primes X are $(e_1e_2e_3e_4)^n e_1e_{10}e_4$. Note that the product of two primes is a prime in this case. The only nonprimes are powers of primes. Note that prime depends on orientation of path. A prime is an equivalence class in the free homotopy class of the graph. So the Ihara zeta function is an infinite product.

$$\zeta(u,X) = \prod_{[p]}(1-u^{\nu(p)})^{-1}$$

This makes sense for $|u| < R_X =$ radius of convergence. In this example we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Then

$$\begin{aligned} &(1-u^2)^1 \det(I - Au + Qu^2) \\ &= \zeta(u,X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3) \end{aligned}$$

For a $(q+1)$ -regular graph (meaning each vertex has degree q), the radius of convergence satisfies

$$R_x = \frac{1}{q}.$$

For $(q+1)$-regular graphs, there are many functional equations. For instance, let

$$\Lambda_X(u) = (1-u^2)^{r-1+\frac{n}{2}}(1-q^2u^2)^{n/2}\zeta(u,X)$$

Then

$$\Lambda_X(u) = (-1)^n\Lambda_X\left(\frac{1}{qu}\right).$$

To make it look like a zeta function, let $u = q^{-s}$ for $s \in \mathbb{C}$, we have

$$F_X(s) := \Lambda_X(q^{-s}) = (-1)^n F_X(1-s).$$

The "Riemann hypothesis" for $(q+1)$-regular graphs is that $\zeta(q^{-s},X)$ has no poles with $0 < Re(s) < 1$ unless $Re(s) = \frac{1}{2}$.

Definition A $(q+1)$-regular graph is called **Ramanujan** if the "Riemann hypothesis" holds.

It is easy to check that all complete graphs on $(q+1)$ graphs are Ramanujan.

An alternate description of Ramanujan graphs:

Let $X = (V,E)$ be a connected, undirected graph. (in this part $k = q+1$) Let $F \subseteq V$, then $\partial F = \left\{e \in V : e \text{ connects } F \text{ to } V-F\right\}$. For example, see the Petersen graph. An analogue of the Cheeger constant of a graph is the **expanding constant** or **isoperimetric constant**.

$$\begin{aligned} h(X) &= \inf\left\{\frac{|\partial F|}{\min\{|F|,|V-F|\}} : F \subseteq V, 0 < |F| < \infty\right\} \\ &\leq \frac{7}{\min\{3,7\}} = \frac{7}{3}. \end{aligned}$$

This constant tells us how efficiently information can be transmitted among vertices. A family of efficient networks X_n should have as few wires as possible (degree k out of each vertex is fixed) and h as large as possible while number of vertices going to ∞. $|V_n| \to \infty$, $h(X_n) > \varepsilon$ for fixed ε and all n.

Complete graphs have large h, but degree is $n - 1$, number of edges grows quadratically.

It turns out that $h(X_n) > \varepsilon$ can be characterized as a spectral property of the adjacency matrices A_n. It turns out that the best family turn out to be the Ramanujan graphs.

Note every k gives a family of Ramanujan graphs.

Theorem Such families exist when $k = p + 1$, p prime, or $k = p^m + 1$.

The first proofs used very complicated number theory, but also a lot of spectral theory.