GEOMETRIC QUANTIZATION

1. The basic idea

The setting of the Hamiltonian version of classical (Newtonian) mechanics is the phase space (position and momentum), which is a symplectic manifold. The typical example of this is the cotangent bundle of a manifold. The manifold is the configuration space (ie set of positions), and the tangent bundle fibers are the momentum vectors. The solutions to Hamilton's equations (this is where the symplectic structure comes in) are the equations of motion of the physical system. The input is the total energy (the Hamiltonian function on the phase space), and the output is the Hamiltonian vector field, whose flow gives the time evolution of the system, say the motion of a particle acted on by certain forces. The Hamiltonian formalism also allows one to easily compute the values of any physical quantities (observables, functions on the phase space such as the Hamiltonian or the formula for angular momentum) using the Hamiltonian function and the symplectic structure.

It turns out that various experiments showed that the Hamiltonian formalism of mechanics is sometimes inadequate, and the highly counter-intuitive quantum mechanical model turned out to produce more correct answers. Quantum mechanics is totally different from classical mechanics. In this model of reality, the position (or momentum) of a particle is an element of a Hilbert space, and an observable is an operator acting on the Hilbert space. The inner product on the Hilbert space provides structure that allows computations to be made, similar to the way the symplectic structure is a computational tool in Hamiltonian mechanics.

What is a quantization? The whole point of it is that the quantum mechanics should agree with classical mechanics to some extent in the situations where classical mechanics is a good approximation to reality. So a quantization is a map from the set of observables of a classical system (ie smooth functions on a symplectic manifold) to the set of observables of a quantum system (ie operators on Hilbert space). This map should satisfy certain properties in order to preserve the physical interpretations. Still, there are many ways to define a quantization. Geometric quantization is one type of quantization, and its aim is to provide an intrinsic construction that is essentially canonical and that is geometrically determined.

2. Hamiltonian Mechanics

2.1. Hamiltonian vector fields and flows.

Let (M,ω) be a symplectic manifold. Given a differentiable function $H : M \to \mathbb{R}$ (called the *Hamiltonian*), there exists a unique vector field X_H such that

$$\begin{aligned} -dH\,(Y) &= \omega\,(X_H, Y)\,, \text{ or} \\ -dH &= i\,(X_H)\,\omega \end{aligned}$$

for every vector field Y on M. (Note: some authors use a negative sign in the equation above.) Such a vector field X_H exists and is unique because the nondegeneracy of ω implies that the linear map

$$i\,(\cdot)\,\omega : T_xM \to T_x^*M$$

has a trivial kernel. In canonical coordinates (ie $\omega = \sum dq_j \wedge dp_j$), we have

$$X_H = \sum -\frac{\partial H}{\partial y_j}\partial_{x_j} + \frac{\partial H}{\partial x_j}\partial_{y_j}.$$

Check:

$$\begin{aligned} i(X_H)\,\omega &= \sum i\left(-\frac{\partial H}{\partial y_j}\partial_{x_j} + \frac{\partial H}{\partial x_j}\partial_{y_j}\right) dx_j \wedge dy_j \\ &= -\sum\left(\frac{\partial H}{\partial y_j}dy_j + \frac{\partial H}{\partial x_j}dx_j\right) \\ &= -dH. \end{aligned}$$

The flow generated by the vector field X_H is called the Hamiltonian flow. That is, for each $t \in (-\varepsilon, \varepsilon)$, let the Hamiltonian flow $\phi_H^t : M \to M$ be the diffeomorphism defined by

$$\frac{\partial}{\partial t}\phi_H^t(x) = X_H\left(\phi_H^t(x)\right);\; \phi_H^0(x) = x.$$

For fixed $x \in M$, $\phi_H^t(x)$ is just an integral curve of X_H with initial point x. Note that the map $f \mapsto X_f$ is a linear map from smooth functions to smooth vector fields. By construction, observe that the function H is constant along the integral curves of X_H (ie H is an *integral* of ϕ_H^t), because

$$\frac{\partial}{\partial t}H\left(\phi_H^t(x)\right) = dH\left(\frac{\partial}{\partial t}\phi_H^t(x)\right) = dH\left(X_H\left(\phi_H^t(x)\right)\right),$$

and

$$dH(X_H) = -\omega(X_H, X_H) = 0$$

since ω is a 2-form.

Observe that each X_H is a symplectic vector field, meaning that

$$\begin{aligned} \mathcal{L}_{X_H}\omega &= (d \circ i(X_H) + i(X_H)\,d)\,\omega \\ &= d\,(i(X_H)\,\omega) = 0. \end{aligned}$$

(True since $i(X_H)\,\omega = -dH$, which is closed.) A consequence of this is that for each $t \in (-\varepsilon, \varepsilon)$, ϕ_H^t is a symplectomorphism, because

$$\frac{\partial}{\partial t}\left(\left(\phi_H^t\right)^*\omega\right) = \left(\phi_H^t\right)^*\left(\mathcal{L}_{X_H}\omega\right) = 0.$$

Another interesting fact: the set of symplectic vector fields forms a Lie subalgebra of the Lie algebra of vector fields, and in fact it is the Lie algebra corresponding to the Lie group of symplectomorphisms. This follows from the fact that if X and Y are symplectic vector fields, then

$$\begin{aligned}
i\left([X,Y]\right)\omega &= i\left(\mathcal{L}_X Y\right)\omega = i\left(\left.\frac{d}{dt}\right|_{t=0}\left(\phi_X^t\right)_* Y\right)\omega \\
&= \left.\frac{d}{dt}\right|_{t=0} i\left(\left(\phi_X^t\right)_* Y\right)\omega \\
&= \left.\frac{d}{dt}\right|_{t=0}\left(\phi_X^t\right)^*\left(i\left(Y\right)\omega\right) = \mathcal{L}_X\left(i\left(Y\right)\omega\right) \\
&= d\circ i\left(X\right)\left(i\left(Y\right)\omega\right) = d\left(\omega\left(Y,X\right)\right),
\end{aligned}$$

so that $i\left([X,Y]\right)\omega$ is also closed.

Note that not every symplectic vector field V is Hamiltonian; V is symplectic iff $i\left(V\right)\omega$ is closed. If in fact $i\left(V\right)\omega$ is exact, then V is Hamiltonian. Thus, the obstruction to a symplectic vector field being Hamiltonian lies in $H^1\left(M\right)$.

Example 2.1. *This is the typical example of a Hamiltonian vector field and flow. Consider the symplectic manifold* $\left(\mathbb{R}^2, \omega = dx\wedge dy\right)$*, with Hamiltonian function* $H\left(x,y\right) = \frac{1}{2}\left(x^2+y^2\right)$*. Then* $dH = xdx + ydy$*, and*

$$i\left(x\partial_y - y\partial_x\right)\omega = -xdx - ydy = -dH,$$

so that the Hamiltonian vector field is

$$X_H = \partial_\theta = \left(x\partial_y - y\partial_x\right).$$

Note that H *is constant on the integral curves of* X_H*, which are circles.*

2.2. **Poisson bracket.** A skew-symmetric, bilinear operation on the differentiable functions on a symplectic manifold M, the *Poisson bracket* $\{f,g\}$, is defined by the formula

$$\{f,g\} = \omega\left(X_f, X_g\right) = -df\left(X_g\right) = -X_g\left(f\right) = X_f\left(g\right),$$

which in canonical coordinates (ie $\omega = \sum dx_j\wedge dy_j$ is

$$\{f,g\} = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j}\frac{\partial g}{\partial x_j}.$$

Note that

$$\{g,f\} = -\{f,g\}$$

Proposition 2.2. *We have*

$$X_{\{f,g\}} = \left[X_f, X_g\right],$$

where the right hand side represents the Lie bracket of vector fields.

Proof.

$$\begin{aligned}
\omega\left(X_{\{f,g\}}, Y\right) &= -Y\left(\{f,g\}\right) \\
&= Y\left(X_g\left(f\right)\right) \\
&= Y\left(df\left(X_g\right)\right)
\end{aligned}$$

and since X_f and X_g are symplectic, by the previous section we have

$$\begin{aligned}\left(i\left(\left[X_f, X_g\right]\right)\omega\right)(Y) &= d\left(\omega\left(X_g, X_f\right)\right)(Y)\\ &= Y\left(\omega\left(X_g, X_f\right)\right)\\ &= -Y\left(\omega\left(X_f, X_g\right)\right)\\ &= Y\left(df\left(X_g\right)\right).\end{aligned}$$

□

As a consequence, the Poisson bracket satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0,$$

which means that the vector space of differential functions on M, endowed with the Poisson bracket, has the structure of a Lie algebra over $\mathbb{R}$, and the assignment $f \mapsto X_f$ is a Lie algebra homomorphism, whose kernel consists of the locally constant functions (constant functions if M is connected). Note that many books (such as McDuff-Salamon) on symplectic geometry define $[X,Y] = \mathcal{L}_Y X = -\mathcal{L}_X Y$ and $i\left(X_H\right)\omega = dH$ so that this still becomes a Lie algebra homomorphism.

A function $g : M \to \mathbb{R}$ is called an *integral* of the flow ϕ_H^t if g is constant on the integral curves of X_H, ie $X_H(g) = 0$. Clearly, H is an integral.

Proposition 2.3. *A function $g : M \to \mathbb{R}$ is an integral of the Hamiltonian flow associated to $H : M \to \mathbb{R}$ if and only if $\{g, H\} = 0$.*

Proof. $X_H(g) = \{H, g\} = -\{g, H\}$. □

2.3. **Relationship to Physics: Hamiltonian mechanics.** The canonical coordinates for the phase space of a mechanical system are $(\mathbf{p}, \mathbf{q}) = (p_1, ..., p_n, q_1, ..., q_n)$, where $\mathbf{q}$ corresponds to position and $\mathbf{p}$ corresponds to momentum. One often thinks of the manifold as the cotangent bundle T^*Q, where Q is the configuration space, and the momentum coordinates live in the fibers. The symplectic form is

$$\omega = \sum dp_j \wedge dq_j.$$

Given a Hamiltonian function H (the total energy!), the Hamiltonian flow is the solution to the system of differential equations

$$\frac{\partial}{\partial t}(\mathbf{q}, \mathbf{p}) = X_H(\mathbf{q}, \mathbf{p}) = \sum -\frac{\partial H}{\partial q_j}\partial_{p_j} + \frac{\partial H}{\partial p_j}\partial_{q_j},$$

which implies

$$\sum \frac{\partial p_j}{\partial t}\partial_{p_j} + \frac{\partial q_j}{\partial t}\partial_{q_j} = X_H = \sum -\frac{\partial H}{\partial q_j}\partial_{p_j} + \frac{\partial H}{\partial p_j}\partial_{q_j},$$

or

$$\begin{aligned}\frac{\partial q_j}{\partial t} &= \frac{\partial H}{\partial p_j}\\ \frac{\partial p_j}{\partial t} &= -\frac{\partial H}{\partial q_j}.\end{aligned}$$

These are know as Hamilton's equations of motion. For example, if

$$\begin{aligned} H &= KE + PE \\ &= \frac{\|p\|^2}{2m} + V(q), \end{aligned}$$

then the equations of the integral curves become

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial t} &= \frac{\mathbf{p}}{m} \\ \frac{\partial \mathbf{p}}{\partial t} &= -\nabla V, \end{aligned}$$

coinciding with the physical laws that

$$\begin{aligned} momentum &= mass * velocity, \\ Force &= -\nabla V = \frac{\partial \mathbf{p}}{\partial t}. \end{aligned}$$

Note that the fact that H is constant along the integral curves is the principle of conservation of energy.

The evolution of any observable $f \in C^\infty$ is given by

$$\frac{\partial f}{\partial t} = X_H f = \sum -\frac{\partial H}{\partial q_j}\partial_{p_j} f + \frac{\partial H}{\partial p_j}\partial_{q_j} f.$$

2.4. **Geodesic flow on the cotangent bundle.** We repeat the notation given previously. Let M be any smooth manifold, and let T^*M denote its cotangent bundle. Then (T^*M, ω) is a symplectic manifold, where the symplectic two-form ω is defined as follows. Since the sections of $\pi : T^*M \to M$ are one-forms, there is a canonical one-form λ on T^*M given by

$$\lambda(\xi_x) = \pi^*(\xi_x),$$

where since $\pi(\xi_x) = x$

$$\pi^* : T_x^*M \to T_{\xi_x}^*(T_x^*M)$$

is the pullback. The symplectic form ω on T^*M is

$$\omega = d\lambda.$$

Suppose that we have chosen a Riemannian metric on M. Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian function defined by

$$H(\xi_x) = \|\xi_x\|^2.$$

The Hamiltonian differential equations obtained are precisely the differential equations of geodesics. Note that this corresponds to H being (up to a constant) kinetic energy only, so it means there are no external potentials (and so no external forces).

2.5. **Completely integrable Hamiltonian systems.** If (M, ω) is a symplectic manifold of dimension $2n$, then it is called *completely integrable* (or simply *integrable*) if there exist n independent Poisson commuting integrals $F_1, ..., F_n$. The word *independent* means that the vectors $\nabla F_1, ..., \nabla F_n$ are linearly independent at each point and that $\{F_j, F_k\} = 0$ for each j, k. Given such a system, the level sets of the $\{F_j\}$ are invariant under the Hamiltonian flows of all n of the functions. Since the functions commute, the compact, connected components of level sets of the form $T_c = \{x \in M : F_j(x) = c_j \text{ for all } j\}$ are n-dimensional tori and are in fact Lagrangian submanifolds (ω vanishes on them). Near such an invariant torus, the symplectic manifold is foliated by such tori, as explained in the next paragraph.

These completely integrable Hamiltonian systems have an extremely simple structure. There are special coordinates called *action-angle variables* that are chosen as follows. In a neighborhood of each invariant torus T_c, the variables $(\xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n$ are defined by

$$\begin{aligned} \eta &= (c_1, ..., c_n), \\ &\xi \text{ is a coordinate on each } T_c \end{aligned}$$

such that the Hamiltonian flow of $H(\xi, \eta)$ (thought of as a function on M) is given by the straight-line solutions of the differential system

$$\dot{\xi} = \frac{\partial H}{\partial \eta}, \ \dot{\eta} = 0.$$

Thus the physical trajectories can be solved completely by doing integrals!

This dynamical behavior of integrable systems is very exceptional, and an arbitrarily small perturbation of the dynamical system will destroy many of the invariant tori. On the other hand, if the frequency vector $\dot{\xi} = \frac{\partial H}{\partial \eta}$ has rationally independent coordinates which satisfy certain Diophantine inequalities and if $\frac{\partial^2 H}{\partial \eta^2}$ is nonsingular, then the corresponding invariant torus survives under sufficiently small perturbations which are sufficiently smooth. This is roughly the content of the Kolmogorov-Arnold-Moser Theorem, from which KAM theory evolved.

3. Quantization

As mentioned earlier, quantization is an assignment

$$\mathcal{Q} : C^{\infty}(M) \rightarrow Op(\mathcal{H})$$

mapping a classical observable (function on phase space (M, ω)) to an operator on Hilbert space $\mathcal{H}$. Following are the rules of **canonical quantization**.

- **Q1:** $\mathcal{Q}$ is $\mathbb{R}$-linear.
- **Q2:** $\mathcal{Q}$ maps the constant function 1 to the identity $\mathbf{1}$ on $\mathcal{H}$.
- **Q3:** $\mathcal{Q}(f)^* = \mathcal{Q}(f)$ for all $f \in C^{\infty}(M)$. (ie real-valued functions should map to operators with real spectrum.
- **Q4:** (Dirac's quantum condition) $[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar\mathcal{Q}(\{f, g\})$. One may think of this as a series of commutation relations defining an algebra.
- **Q5:** (irreducibility condition) If $\{f_1, ..., f_k\}$ is a complete set of observables, then $\{\mathcal{Q}(f_1), ..., \mathcal{Q}(f_k)\}$ is a complete set of operators (meaning the only set of operators that commute with all of them are multiples of the identity). We are basically finding an irreducible representation of the algebra defined by (Q1) through (Q4).

Unfortunately, it turns out that it is in general not possible for the quantization to satisfy both (Q4) and (Q5), and there are many types of quantizations providing a nice compromise. We will only require that (Q4) hold for a sufficiently complete set of observables. Geometric quantization is a procedure that addresses the question of existence and classification of quantizations by slightly changing the rules of the game.

Example: Let $Q = \mathbb{R}^n$, $M = T^*Q$ with the standard symplectic form $\omega = \sum dp_j \wedge dq_j$. Note that the coordinate functions q^k and p_l form a complete set of observables. According to (Q4) above, we require:

$$\begin{aligned}\left[\mathcal{Q}\left(q^{k}\right),\mathcal{Q}\left(q^{l}\right)\right] &= \left[\mathcal{Q}\left(p_{k}\right),\mathcal{Q}\left(p_{l}\right)\right]=0,\\ \left[\mathcal{Q}\left(q^{k}\right),\mathcal{Q}\left(p_{l}\right)\right] &= i\hbar\delta^{k}_{\ l}\ .\end{aligned}$$

Using

$$\{f,g\}=\sum_{j=1}^{n}\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}-\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j}.$$

This means that these quantum observables form the *Heissenberg algebra*, and (Q5) essentially means that we need to find an irreducible representation of this algebra. By the Stone - von Neumann Theorem, any such representation (that exponentiates to a representation of the Heissenberg group) is unitarily equivalent to $L^2(Q)=L^2(\mathbb{R}^n)$ with

$$\begin{aligned}\mathcal{Q}\left(q^{k}\right)\psi(x) &= x^{k}\psi(x)\\ \mathcal{Q}\left(p_{l}\right)\psi(x) &= -i\hbar\frac{\partial\psi}{\partial x^{l}}(x).\end{aligned}$$

(This is the famous Schrödinger representation.) Note that the fact that $L^2(Q)$ is used as the Hilbert space is not an axiom of the quantum mechanics (ie position replaced by wave function) but a consequence of the representation theory.

What are the quantizations of other observables? Kinetic energy is $p^2=\sum p_l p_l$ and should be represented by

$$\mathcal{Q}\left(p^2\right)=\hbar^2\Delta.$$

We certainly need quantities like this for the quantum physics. But note that $\mathcal{Q}(fg)\neq\mathcal{Q}(f)\mathcal{Q}(g)$ in general. By imposing (Q3) or (Q4) we see that

$$\mathcal{Q}\left(p_l q^k\right)=\frac{1}{2}\left(\mathcal{Q}(p_l)\mathcal{Q}\left(q^k\right)+\mathcal{Q}\left(q^k\right)\mathcal{Q}(p_l)\right).$$

It turns out that the quadratic observables form a Lie algebra under Poisson brackets, and that Lie algebra is isometric to $\mathfrak{sp}(n)$. In general it is desirable that when we quantize a symplectic vector space, we should obtain a representation of the symplectic Lie algebra. We have now been able to quantize all the linear observables and quadratic observables (much more than just the complete set of observables) in a consistent way.

Unfortunately, we are unable to quantize cubic observables (conflict with (Q4)). Thus, even in the simplest case, full quantization is not possible.

4. Geometric Quantization

Geometric quantization (GQ) is established in two or more steps. The first step is prequantization. The second step involves something called polarization.

4.1. Prequantization.

Prequantization is the construction of an assignment $\mathcal{Q}$ that satisfies (Q1) through (Q4). The first attempt of a map from the Poisson algebra $C^\infty(M,\omega)$ to operators on Hilbert space is

$$f\mapsto -i\hbar X_f,$$

where the Hilbert space is some closure of $C^\infty(M)$. This map satisfies (Q1), (Q3), and (Q4), but (Q2) fails because any constant function gets mapped to the zero vector field.

After a little experimentation, one can show that if $M = T^*Q$, then the assignment $\mathcal{P} : C^\infty(M,\omega) \to Op(L^2(M,\omega))$

$$\mathcal{P}(f) = -i\hbar X_f - \theta(X_f) + f$$

satisfies (Q1) through (Q4), where θ is the canonical one-form on the cotangent bundle $T^*Q \xrightarrow{\pi} Q$ such that $\omega = d\theta$. (Recall: if $v_y \in T_y T^*Q$, then $\theta(v_y) = y(\pi_* v_y)$.) Note that $\mathcal{P}(f)$ is a linear differential operator on $L^2(M,\omega)$, and this is a faithful representation of the Poisson algebra. If $Q = \mathbb{R}^n$, then this map is

$$\begin{aligned} \mathcal{P}\left(q^k\right) &= i\hbar\partial_{p_k} + q_k \\ \mathcal{P}(p_l) &= -i\hbar\partial_{q_l} \end{aligned}$$

This quantization is apparently equivalent to the Schrödinger quantization/representation, but only acting on functions of $\mathbf{q}$ alone. Note that (Q5) fails, because ∂_{p_k} commutes with all $\mathcal{P}\left(q^k\right)$ and $\mathcal{P}(p_l)$. Also, it fails to produce the second order operators associated to quadratic observables in $\{p_l\}$, such as kinetic energy. How would we fix this problem? GQ can remedy this, but unfortunately that means that the appropriate $\mathcal{P}(p^2)$ does not act merely on the space of functions that depends on the coordinates alone. It changes the representation space (or, the **polarization**). To associate an operator to p^2, one must compensate for the change in polarization caused by the flow of p^2. This is an analytically difficult procedure based on BKS (Blattner-Kostant-Sternberg) kernels. The understa nding of this is now somewhat incomplete.

Another problem with this quantization is that θ could be replaced by $\theta + dF$ with F a function on M. This can be described as conjugating by the phase factor $e^{-iF/\hbar}$ in $L^2(M,\omega)$, so the resulting representations are unitarily equivalent. Since the F is only determined up to a constant, there is a phase ambiguity in the prequantum phase functions. So this suggests that instead of functions we should use sections of a line bundle.

What you look for is a connection D on a line bundle $L \approx M \times \mathbb{C}$ that in a particular trivialization looks like

$$D = d - \frac{i}{\hbar}\theta,$$

where θ is the canonical one-form on the cotangent bundle $M = T^*Q$. Then we would want our prequantum operator $\mathcal{P}(f)$ to look like

$$\mathcal{P}(f) = -i\hbar D_{X_f} + f.$$

Another way to look at $\mathcal{P}(f)$ is as follows. Let

$$\begin{aligned} \mathcal{L}_f &= \theta(X_f) - f \\ &= p_k \frac{\partial f}{\partial p_k} - f \end{aligned}$$

be the Lagrangian of f. Then

$$\begin{aligned} \mathcal{P}(f) &= -i\hbar X_f - \mathcal{L}_f \\ &= -i\hbar \frac{d}{dt}\left(\widehat{\Phi}_t^f \psi\right)\Big|_{t=0}, \end{aligned}$$

where

$$\left(\widehat{\Phi}_t^f \psi\right)(m) = \psi\left(\Phi_t^f(m)\right) \exp\left(-\frac{i}{\hbar}\int_0^t \mathcal{L}_f\left(\Phi_{t'}^f(m)\right) dt'\right),$$

where $\Phi^f_{t'}$ is the Hamiltonian flow, ψ is a section of L.

This eventually leads to the following definition. Note that the curvature Ω of L is defined by

$$\begin{aligned}\Omega(X,Y) &= i\left([D_X, D_Y] - D_{[X,Y]}\right)\\ \Omega &= iD^2 = \frac{1}{\hbar}d\theta = \frac{1}{\hbar}\omega\end{aligned}$$

Definition 4.1. *A* ***prequantization*** *of a symplectic manifold (M,ω) is a pair (L,D), where L is a complex Hermitian line bundle over M and D is a compatible connection with curvature $\frac{1}{\hbar}\omega$. The* ***prequantum Hilbert space*** *is the completion of the space of smooth sections of L in $L^2(M,L)$, where the L^2 metric is defined with respect to the Liouville measure ω^n on M and Hermitian structure on the fibers.*

Important Note: We can do this **only** if ω satisfies the *Bohr-Sommerfeld condition*, namely that $\frac{1}{2\pi\hbar}\omega$ defines an integral cohomology class. Topologically, line bundles are classified by the first Chern class $c_1(L) \in H^2(M, 2\pi\mathbb{Z})$. In de Rham cohomology, this is represented by the curvature of any connection on L. This condition is both necessary and sufficient for the prequantization to work. In that case, we call (M,ω) **quantizable**. Another way to state the condition is that the integral

$$\int_S \frac{1}{2\pi\hbar}\omega$$

is an integer, for any choice of closed,oriented 2-dimensional submanifold S of M.

Note that if ω is exact (eg $M = T^*Q$), then the condition is clearly satisfied, and since the cohomology class is trivial, L is also trivial. Sometimes, if (M,ω) is not quantizable, one may rescale the symplectic form in some way to make the condition work.

How would we classify prequantizations of a quantizable symplectic manifold (M,ω)? We will say that two prequantizations (L,D) and (L',D') are equivalent if there exists a bundle isomorphism $F : L \to L'$ such that $F^*D' = D$. Note that if (L,D) is a prequantization, then $(L\otimes L_0, D\otimes D_0)$ is also a prequantization, for any flat line bundle (L_0, D_0), because the curvature of a tensor product is the sum of the curvatures. Conversely, it turns out that if (L,D) and (L',D') are equivalent, then

$$\begin{aligned}(L',D') &\sim (L',D')\otimes(L,D)\otimes(L^*,D^*)\\ &\sim (L,D)\otimes\left[(L',D')\otimes(L^*,D^*)\right],\end{aligned}$$

and $[(L',D')\otimes(L^*,D^*)]$ is flat. Thus the classification of prequantizations amounts to the classification of flat line bundles (and is, in fact, independent of ω). And this is standard: the isomorphism classes of flat line bundles over M is in one-to-one correspondence with elements of $H^1(M, U(1))$, or $\mathrm{Hom}(\pi_1(M), U(1))$. Note that the freedom of choosing (L,D) comes in two varieties. First, one may fix a certain line bundle and may look at the space of nonequivalent flat connections on it. This space is parametrized by

$$H^1(M,\mathbb{R}) \diagup H^1(M,\mathbb{Z}) \sim U(1)^{b_1(M)},$$

where $b_1(M) = \dim H^1(M,\mathbb{R})$ is the first Betti number. In other words, D and $D+\alpha$ are inequivalent if and only if α is neither integral nor exact. Second, one may look at topologically inequivalent line bundles, which are classified by the first Chern class in $H^2(M,\mathbb{Z})$.

5. Polarizations

The problem with the prequantization is that it is too big, and we need to cut down the space of sections in some way.

Definition 5.1. *A* ***polarization*** *of (M,ω) is a smooth distribution of the complexified tangent space $T_{\mathbb{C}}M$, in other words a smooth choice of subspace of $T_{\mathbb{C},x}M$ at each $x \in M$, such that*

1. *P_x is Lagrangian (maximally isotropic) for each $x \in M$. Recall: this means $i\,(v_x)\,\omega_{\mathbb{C}} = 0$ for all $v_x \in P_x$, and there is no larger subspace that has this property and contains P_x. In particular, this means that the dimension of P_x is half the dimension of M.*
2. *P is integrable, meaning that $[\Gamma(P), \Gamma(P)] \subset \Gamma(P)$.*

The simplest kind of polarization is called a **real polarization**, where P is the complexification of a distribution $P' \subset TM$ that is involutive and thus is the tangent bundle of a Lagrangian foliation of M. If $M = T^*Q$, the vertical vectors produce a real polarization.

One then completes the quantization procedure by restricting to sections of L that are covariantly constant in the P directions. In the **real quantization** case, these are the basic sections of L with respect to the Lagrangian foliation of M.

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