A geometric glance at zeta functions, L-functions, and automorphic forms

(Ken Richardson)

Example: the Riemann zeta function

Consider the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

a formula which is valid on the open half-plane $Re(s) > 1$; it converges absolutely and uniformly on compact subsets of this half-plane. Note that

$$2\zeta(2s) = \sum_{n\neq 0} (n^2)^{-s}, Re(s) > \frac{1}{2},$$

and the numbers n^2, $n \in \mathbb{Z}$, are precisely the eigenvalues of $\Delta = -\frac{d^2}{d\theta^2}$ corresponding to eigenfunctions $e^{in\theta}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Thus, these numbers are the eigenvalues $\lambda = n^2$ of the Laplacian Δ on the Hilbert space of L^2 functions on the unit circle. Consider each individual term $(n^2)^{-s}$. We have the following formula.

Lemma $\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \exp(-\lambda t)\, dt$ for $\lambda > 0$.

Proof Using the substitution $u = \lambda t$, we have

$$\int_0^{\infty} t^{s-1} \exp(-\lambda t)\, dt = \lambda^{-s} \int_0^{\infty} u^{s-1} \exp(-u)\, du = \lambda^{-s}\Gamma(s).$$

Thus, we have

$$\sum_{\lambda} \lambda^{-s} = \frac{1}{\Gamma(s)} \sum_{\lambda} \int_0^{\infty} t^{s-1} \exp(-\lambda t)\, dt$$
$$= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{\lambda} \exp(-\lambda t) \right) dt$$

by Tonelli's theorem. We have just shown that

$$2\zeta(2s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{n\neq 0} \exp(-n^2 t) \right) dt.$$

The heat operator

The function $K(t) = \sum_{n\in Z} \exp(-n^2 t)$ is the trace of the heat operator on the circle. The heat operator is the operator $K_t : L^2(S^1) \to L^2(S^1)$ that depends on t that satisfies

$$K_t(e^{in\theta}) = e^{-n^2 t} e^{in\theta} = (\exp(-t\Delta)) e^{in\theta}.$$

It is clear that its trace is $\sum_{n\in Z} \exp(-n^2 t)$, but why is K_t called the heat operator? In physical terms, the operator inputs a temperature function on the circle, and the output is the temperature at time t, as governed by the heat equation. On a Riemannian manifold M, the heat equation for an unknown function $u(x,t)$ with $t > 0$ (time) and $x \in M$ (position)

$$(\partial_t + \Delta_x)u(x,t) = 0,$$

and the initial value problem for the heat equation is

$$\begin{aligned}(\partial_t + \Delta_x)u(x,t) &= 0,\\ u(x,0) &= f(x),\end{aligned}$$

and you can think of $f(x)$ as the initial temperature distribution and $u(x,t)$ the temperature at time t. The heat operator K_t satisfies

$$\begin{aligned}(\partial_t + \Delta_x)(K_t(f)(x)) &= 0,\\ \lim_{t\to 0^+} K_t(f)(x) &= f(x).\end{aligned}$$

Note that if we take the special case of $f(\theta) = e^{in\theta}$, then $u(\theta,t) = e^{-n^2 t}e^{in\theta}$ satisfies the initial value problem for the heat equation on the circle:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \theta^2}\right)u(\theta,t) &= 0,\\ u(\theta,0) &= e^{in\theta}.\end{aligned}$$

There is a formula for the heat operator. This formula involves the heat kernel $K(t,x,y)$, with $t > 0$, $x,y \in M$. One may think of this as the amount of heat at time t at position x, if at time 0 a delta-function heat distribution covers the manifold, with singularity at y (that is, if all the heat is concentrated at the point y). Anyway, the formula is

$$u(x,t) = K_t(f)(x) = \int_M K(t,x,y)f(y)\,\mathrm{dvol}_y\,,$$

and $K(t,x,y)$ satisfies the formula

$$\begin{aligned}(\partial_t + \Delta_x)K(t,x,y) &= 0,\\ \lim_{t\to 0^+} K(t,x,y) &= \delta(x,y),\end{aligned}$$

where $\delta(x,y)$ is the Dirac delta distribution. Under fairly weak assumptions (eg Ricci curvature bounded from below by a negative constant and $K(t,x,y) \to 0$ as $x \to \infty$), the kernel $K(t,x,y)$ is unique. If M is Euclidean space $\mathbb{R}^n$, $K(t,x,y)$ is given by

$$K(t,x,y) = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{4t}\right).$$

If we let $\{\lambda\}$ be the eigenvalues of the Laplacian Δ and $\{\phi_\lambda\}$ the corresponding L^2-orthonormal basis of eigenfunctions, we have the following formula for $K(t,x,y)$ that can be easily verified:

$$K(t,x,y) = \sum_\lambda \exp(-t\lambda)\phi_\lambda(x)\overline{\phi_\lambda(y)},$$

where in the formula the eigenvalues are repeated according to multiplicities. Note that the sum converges absolutely for $t > 0$ (and uniformly if M is compact). Note that the trace of the heat operator is

$$Tr(K_t) = \int_M K(t,x,x)\,\mathrm{dvol}_x = \sum_\lambda \exp(-t\lambda).$$

The heat kernel on a general Riemannian manifold differs from that on Euclidean space, but for small time $t > 0$, the two are similar. In fact, the following asymptotic formula for the heat kernel on a compact manifold M was shown by Minakshisundaram and Pleijel in 1949:

Theorem (Minakshisundaram and Pleijel, 1949) For every positive integer k,

$$K(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{r^2}{4t}\right)\left(u_0(x,y) + tu_1(x,y) + t^2 u_2(x,y) + \ldots + t^k u_k(x,y) + O(t^{k+1})\right),$$

where r is the distance between x and y in M, and each $u_k(x,y)$ is a smooth function of x,y that only depends on the metric and its covariant derivatives along the minimal geodesic connecting x and y. (It is assumed that x is not in the cut locus of y, so that there is a unique minimal geodesic.) Further, $u_0(x,x) = 1$.

One may actually solve for the coefficient functions $u_j(x,y)$ by plugging the asymptotic formula into the heat equation. From this formula, one may deduce the following asymptotic formula for the trace of the heat operator:

$$Tr(K_t) = \sum_\lambda \exp(-t\lambda) = \frac{1}{(4\pi t)^{n/2}}\left(\mathrm{vol}(M) + tU_1 + t^2 U_2 + \ldots + t^k U_k + O(t^{k+1})\right),$$

where each U_j is an integral of a local quantity dependent on the metric and its covariant derivatives. Moreover, these quantities may be written in terms of curvatures; for instance,

$$U_1 = \frac{1}{6}\int_M R\,\mathrm{dvol},$$

where R is the scalar curvature. In the case of the unit circle, the formula takes the form

$$Tr(K_t) = 1 + 2\sum_{n=1}^{\infty} \exp(-tn^2) = \frac{1}{(4\pi t)^{1/2}}\left(2\pi + O(t^{k+1})\right) = \frac{\sqrt{\pi}}{\sqrt{t}} + O(t^\kappa)$$

for every $\kappa = k + \frac{1}{2} \geq 0$. The local formula for $K(t,\theta_1,\theta_2)$ is

$$K(t,\theta_1,\theta_2) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{in(\theta_1-\theta_2)} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos(n(\theta_1-\theta_2))$$
$$= \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{(\theta_1-\theta_2)^2}{4t}\right)\left(1 + O(t^{k+1})\right).$$

In the particular case where there is a Riemannian covering $N \xrightarrow{p} M$, we have the formula

$$K_M(t,x,y) = \sum_\gamma K_N(t,\hat{x},\gamma\hat{y}),$$

where $\hat{x} \in p^{-1}(x)$, $\hat{y} \in p^{-1}(y)$, and the sum is over all deck transformations γ. The reason this formula works is that the local expressions for the Laplacian of N and of M are the same, and both sides satisfy the initial value problem on M. The right hand side turns out to be equivariant with respect to the deck transformations, which are isometries. If one thinks about initial heat distributions, the formulas make sense. That is, if the initial temperature distribution on M is a delta function at y, this is equivalent to an initial temperature distribution on N that is the sum of delta functions at each point of $p^{-1}(y)$.

A particular case of the formula above is again on the unit circle, where $\mathbb{R} \to S^1$ is the covering map. The deck transformations are additions of multiples of 2π. We have

$$K(t,\theta_1,\theta_2) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos(n(\theta_1-\theta_2)) = \sum_{m\in\mathbb{Z}} \frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{|(\theta_1-\theta_2-2\pi m)|^2}{4t}\right).$$

Letting $\theta_1 = \theta_2$, we have

$$\begin{aligned}
K(t,\theta,\theta) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-n^2 t} \\
&= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t} \\
&= \frac{1}{(4\pi t)^{1/2}}\sum_{m\in\mathbb{Z}} \exp\left(-\frac{\pi^2 m^2}{t}\right) \\
&= \frac{1}{2(\pi t)^{1/2}} + \frac{1}{(\pi t)^{1/2}}\sum_{m=1}^{\infty} \exp\left(-\frac{\pi^2 m^2}{t}\right).
\end{aligned}$$

This formula is also known as the Jacobi transformation formula for the theta function. Note that when $t = \pi$, both sides of the equation are identical.

Functional equation and analytic continuation of the zeta function

We now apply the heat kernel information to the Riemann zeta function ζ. For $Re(s) > \frac{1}{2}$,

$$\begin{aligned}
2\zeta(2s) &= \sum_{n\neq 0}(n^2)^{-s} \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\left(\sum_{n\in\mathbb{Z}} \exp(-n^2 t) - 1\right)dt \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\left(Tr(K_t) - 1\right)dt \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\left(2\pi K(t,\theta,\theta) - 1\right)dt \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\left(\frac{\sqrt{\pi}}{\sqrt{t}}\sum_{m\in\mathbb{Z}} \exp\left(-\frac{\pi^2 m^2}{t}\right) - 1\right)dt.
\end{aligned}$$

With this in mind, we have the following:

$$2\zeta(2s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(Tr(K_t)-1)dt = \frac{1}{\Gamma(s)}\int_0^\pi t^{s-1}\left(\frac{\sqrt{\pi}}{\sqrt{t}}-1\right)dt$$
$$+\frac{1}{\Gamma(s)}\int_0^\pi t^{s-1}\left(Tr(K_t)-\frac{\sqrt{\pi}}{\sqrt{t}}\right)dt+\frac{1}{\Gamma(s)}\int_\pi^\infty t^{s-1}\left(2\sum_{n=1}^\infty e^{-n^2 t}\right)dt$$

Letting $Tr(K_t) = \frac{\sqrt{\pi}}{\sqrt{t}}\sum_{m\in\mathbb{Z}}\exp\left(-\frac{\pi^2 m^2}{t}\right)$ and $t = \frac{\pi}{u}$ in the second integral and $t = \pi u$ in the third integral, we obtain

$$2\zeta(2s) = \frac{\pi^s}{\Gamma(s)}\left(\frac{1}{s-\frac{1}{2}}-\frac{1}{s}\right)+\frac{\pi^{s-1}}{\Gamma(s)}\int_1^\infty u^{(1-s)}\frac{\sqrt{\pi}\sqrt{u}}{\sqrt{\pi}}\left(2\sum_{k=1}^\infty e^{-k^2\pi u}\right)\frac{\pi}{u^2}du$$
$$+\frac{\pi^s}{\Gamma(s)}\int_1^\infty u^{s-1}\left(2\sum_{n=1}^\infty e^{-n^2\pi u}\right)du$$
$$=\frac{\pi^s}{\Gamma(s)}\left(\frac{1}{s-\frac{1}{2}}-\frac{1}{s}\right)+\frac{\pi^s}{\Gamma(s)}\int_1^\infty u^{(-s-\frac{1}{2})}\left(2\sum_{k=1}^\infty e^{-k^2\pi u}\right)du$$
$$+\frac{\pi^s}{\Gamma(s)}\int_1^\infty u^{s-1}\left(2\sum_{n=1}^\infty e^{-n^2\pi u}\right)du$$
$$=\frac{\pi^s}{\Gamma(s)}\left(-\frac{1}{\frac{1}{2}-s}-\frac{1}{s}\right)+\frac{\pi^s}{\Gamma(s)}\int_1^\infty \frac{1}{u}\left(u^s+u^{(\frac{1}{2}-s)}\right)\left(2\sum_{k=1}^\infty e^{-k^2\pi u}\right)du$$

Thus, we have

$$\xi(s) := \pi^{-s}\Gamma(s)2\zeta(2s)$$

satisfies the functional equation

$$\xi\left(\frac{1}{2}-s\right) = \xi(s)$$
$$= -\frac{1}{\frac{1}{2}-s}-\frac{1}{s}+\int_1^\infty \frac{1}{u}\left(u^s+u^{(\frac{1}{2}-s)}\right)\left(2\sum_{k\in\mathbb{N}} e^{-k^2\pi u}\right)du.$$

Also, observe that the latter formula gives a well-defined formula for $\xi(s)$ that is analytic except with simple poles at $s = 0$ and $s = \frac{1}{2}$. Thus, the formula also gives the analytic continuation formula for $\zeta(s)$ via

$$\zeta(s) = \frac{\pi^{s/2}}{2\Gamma\left(\frac{s}{2}\right)}\xi\left(\frac{s}{2}\right).$$

Note that the standard functional equation for the Riemann zeta function is equivalent to ours above. The key ingredient needed to produce the functional equation is the fact about heat kernels for the cover $\mathbb{R}\to S^1$. But there is a more general approach that yields the analytic continuation in more generality.

Next time:

Analytic continuation for more general zeta functions

Comparing zeta functions of number theory to geometric zeta functions

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