A Mathematical Formalism to Bridge Between Category Theory and Engineering

Henson Graves

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Category theory is often thought to be 'abstract nonsense', but it can be very useful in engineering

Case Study Result From a Project called Algos

Goals

- A mathematical formalism that can be implemented within a proof assistant
- Used for formalizing theories in mathematics, science,engineering, and ...

Technical approach

- An axiomatic topos language (tierney1972sheaf) and sequent calculus deduction system (martin1975intuitionistic)
- Meta Algos is a category meta and is a specification for the proof assistant

Results

- language and deduction system correspond to methods of test and verification in physical environments
- Category constructions (limit cones, sheaves,...) to represent interacting physical systems in the Assistant

Project Trajectory

Stanford Computer Lab (smith1975computer)

- Worked on proof assistant for a formal set theory. Simple proofs were very difficult. Colleague suggested axiomatic topos theory instead of set theory

Algos Project at San Jose State U (graves1985algorith)

- Proof of Concept: Partial implementation of topos axioms
- Homemade deduction system worked
- Equal terms reduced to canonical form where possible

Aerospace Experience

- Enabled solving problem concerning topos axioms
- Enabled developing full mathematical formalism

Initial Implementation Partial

A first order language with **Map** *and* **Type** sorts with variables
Predicates

$$f = g, X = Y \tag{1}$$
$$f : A \to B \equiv Dom(f) = A \; and \; CODom(f) = B \tag{2}$$

Types (Objects)

$$\emptyset, One, X \times Y, X + Y, Y^X, Pow(X), \Omega \tag{3}$$

Some map constructions

- $true : One \to \Omega, \; false : One \to \Omega$
- $a : A = a : One \to A$
- $\rho 1 : X \to X + Y, \rho 2 : X \to X + Y$
- $f : X \to D, g : Y \to D, \Rightarrow f + g : X + Y \to D$
- $f : A \times B \to C \Rightarrow \lambda x.f : A \to C^B,$
- $g : A \to B^T \Rightarrow .g[t] : A \times T \to T$

Constructive Product Axioms

Problem is to find axioms that do not have existential quantifiers. Rule axioms were known for Products, Sums, Exponentials, For example, subtype classification

$$\textbf{tuple}$$

$$a : X \to A, b : X \to B \Rightarrow (a, b) : X \to A \times B \tag{4}$$

$$\textbf{projections}$$

$$f : C \to A, g : C \to B, \Rightarrow \pi 1(f, g) = f, \pi 2(f, g) = g \tag{5}$$

$$(t1, t2) = (r1, r2) \Rightarrow t1 = r1, t2 = r2 \tag{6}$$

$$(\pi 1_{X,Y}, \pi 2_{X,Y}) = id_{(X,Y)} \tag{7}$$

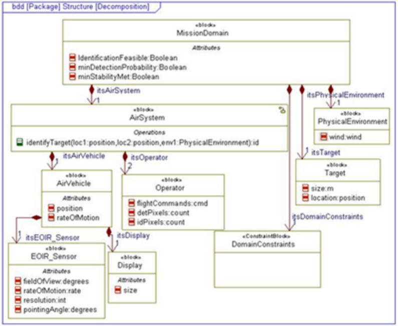
$$(f, g)(h) = (f(h), g(h)) \tag{8}$$

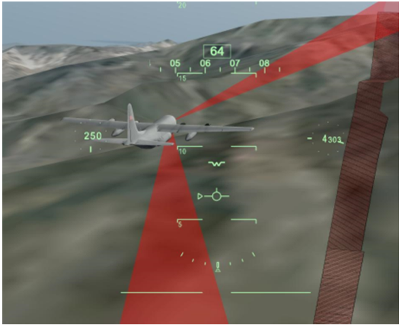
$$\pi : A \times B \to A \Rightarrow Epic(\pi) \tag{9}$$

Projection maps can be renamed.

$$(B \times C) = (\pi 1 : B + \pi 2 : C) = (x : B + y : C) \tag{10}$$

Constructed digital models and simulation and using them for analysis had a big impact on Algos



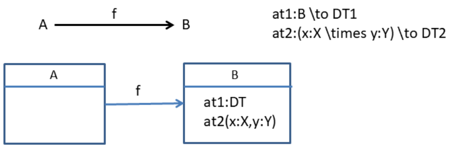


Impact on Algos

- Modeling Language constructions are similar to those of topos theory
- The graphically oriented language of SysML can be adapted for Algos in addition to linear syntax
- Experience enabled completing topos axioms
- Physical test and valuation uses constructive deduction as well as direct physical observation
- Engineering modeling language authoring systems are well developed, easy to use, and scale up
- Model development systems provide conception for their formalization

Adapted Graphically Oriented Syntax From Modeling Languages

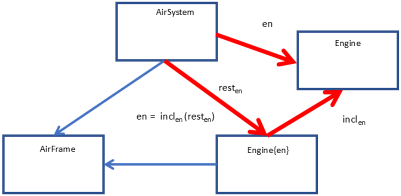
- maps = associations, types = blocks
- strongly overlapping constructions
- graphics can codify many additional assumptions



- an attribute such as $at1$ is a map $at1 : B \to DT$
- many other conventions make for ease of use

Completing Topos Axioms for Algos

First add inclusion subtypes with inclusion and restriction map and establish minimal factorization



Inclusion and Restriction Maps

$$f : A \to B \Rightarrow$$

$$incl_f : B\{f\} \to B, Monic(incl_f), rest_f : A \to B\{f\}, Epic(rest_f) \quad (11)$$

Takes more work to establish factorization is minimal

Empirical Test and Evaluation Uses Constructive Inference as Well as Direct Observation

Formalized deductions are labeled sequents $F : P(\hat{x}) \vdash Q(\hat{x})$. They form a category where the types are formulas and the maps are the labeled sequents. This is a Howard-Curry-Lambek category
The inference rules are presented in a numerator-denominator form where both numerator and denominator are deductions.

$$\frac{F : \Gamma \vdash P \quad G : \Gamma \vdash Q}{(F, G) : \Gamma \vdash P \wedge Q} \tag{12}$$

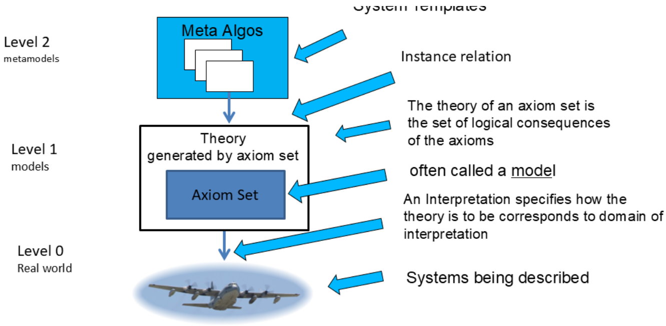
Deductions are equivalent to formulas, for example

$$\frac{F : \Gamma, P \vdash Q}{\lambda P.F : \Gamma \vdash P \Rightarrow Q}. \tag{13}$$

A theorem is a deduction of the form **One** $\vdash$ **F**

Engineering Authoring Assistants Suggest that an Algos Theory Could be the specification for an Implementable

Variations of this diagram are common in engineering and computer science.



- Level 0 is the interpretation domain for a theory
- Level 1 is a theory, i.e, the deductive closure of an axiom set with respect to the deduction system
- Level 3 is the meta theory with a meta types (template) for the theories

Meta Algos is a specification for a proof assistant for categories

- Meta Algos is an Algos theory with additional axioms used to represent axiom set theories, category theories, deductions, functors,...
- Since Meta Algos is a topos and is used to represent categories we distinguish

$$\mathbf{F} :: \mathbf{A} \to \mathbf{B} \ , \ \mathcal{D} :: \mathbf{A} \Rightarrow \mathbf{F}(\mathcal{D}) :: \mathbf{B} \tag{14}$$

- Types are templates, their instances are individual theories
- Meta Algos has additional axioms to make the types into 'sets'

$$\mathbf{\Omega} = \{\mathbf{True}, \mathbf{False}\} \tag{15}$$

$$\mathcal{A}, \mathcal{B} \in \mathbf{A}, \mathcal{B} \in \mathbf{A}\{\mathcal{B}\} \Rightarrow \mathcal{A} = \mathcal{B} \tag{16}$$

- Meta theory makes extensive use of membership, inclusion, and the Algos constructions, including recursion

Axiom Sets, Languages, theories are Recursive Types

Example meta types that have to be defined

- $\textbf{Signature} = \textbf{Const} + \textbf{Pred} + \textbf{Function}$
- $\textbf{Formula}(\textbf{Signature})$
- $\textbf{AxSet} \sqsubseteq \textbf{Formulas}(\textbf{Signature})$
- $\textbf{Deduction}(\textbf{Formulas}(\textbf{Signature}))$
- $(\textbf{F} : \textbf{One} \vdash \textbf{Q}) \in \textbf{Deduction} \Rightarrow \textbf{F} \in \textbf{Theorem}(\textbf{AxSet})$
- $\mathcal{C} = \{A, B, C, D, p1 : A \to B, p2 : A \to B, p3 : B \to C, p3(p2) : A \to C\}$ implies $\Rightarrow \mathcal{C} \in \textbf{Cat}$

Recursive Types in Meta Algos

List of a type in computer science is often written as:

$$\mathbf{List}(\mathbf{A}) = \mathbf{Nil} + \mathbf{Cons}(\mathbf{First}, \mathbf{Rest}) \tag{17}$$

with some axioms. We do the same by renaming of inclusions and projections maps in sums and products.

$$\mathbf{List}(\mathbf{A}) = \emptyset + (\mathbf{A} \times \mathbf{List}(\mathbf{A}) \tag{18}$$

$$\mathbf{List}(\mathbf{A}) = \mathbf{Nil} : \emptyset + \mathbf{Cons} : (\mathbf{A} \times \mathbf{List}(\mathbf{A})) \tag{19}$$

where nil and cons are the names of the inclusion maps.

$$\pi\mathbf{1} : \mathbf{A} \times \mathbf{List}(\mathbf{A}) = \mathbf{A} \ , \ \pi\mathbf{2} : \mathbf{A} \times \mathbf{List}(\mathbf{A}) = \mathbf{List}(\mathbf{A}) \tag{20}$$

imply

$$\mathbf{List}(\mathbf{A}) = \sigma\mathbf{1} : \emptyset + \sigma\mathbf{2} : (\pi\mathbf{1}, \pi\mathbf{2}) \tag{21}$$

or

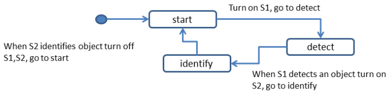
$$\mathbf{List}(\mathbf{A}) = \mathbf{Nil} + \mathbf{Cons}(\mathbf{First}, \mathbf{Rest}) \tag{22}$$

Behavior makes use of abstraction in Meta Algos

Behavior in engineering models is often represented by state machines. A state machine monitors sensors, when they change, depending on the current state, the action transitions to a new state and performs the action to modify the output attribute.

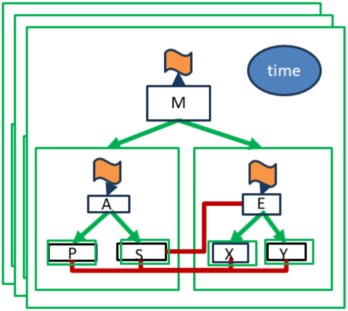
$$M = (In \times State \times Out) \tag{23}$$

and a type T finite linearly ordered time, and a state chart map $d : M \to M$ represents the diagram.



Behavior is represented as an exponential $\mathbf{M}^{\mathbf{Time}}$

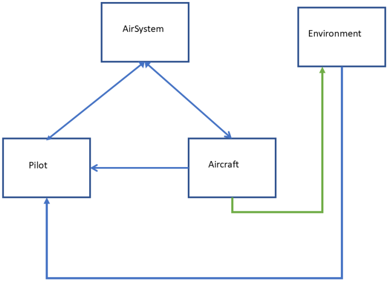
Interacting Systems within $\mathcal{A}$ in Meta Algos



- systems and their subsystems (green boxes)
- a system has a largest element, flags are attributes of largest element
- interaction between systems (red lines)

A System in Meta Algos

A system is a hierarchical structure of maps and types in $\mathcal{A}$ in Meta Algos



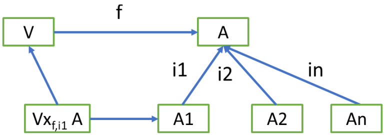
- the system S has a top element
- the system decomposition maps are bidirectional
- the components of S may have arrows to other maps in $\mathcal{A}$
- S is a bi-cone. This enables routing the interfaces of the system component through the greatest type. We write this as $S(A)$ where A is the greatest type of

A a collection of systems in $\mathcal{A}$ is a Covering Sieve one Shows, for example

The pullback of a map $f : V \to A_j$ along S(A) is

$$f^*(S) = \{V \times_{f,i} A_i \to V\} \in S(V) \tag{24}$$

for $A_i \in S(A)$ is a base change.



Showing that a covering by subsystem of $\mathcal{A}$ is a Grothendieck topology requires showing that $\mathcal{A}$ is a Grothendieck topos

Next Steps

- find an open source SysMl development platform
- implement Meta Algos as the proof assistant within the platform
- Build a business plan