How is the subject of formal metrics related to foliation theory?

Lemma: Let ω be a closed p-form on M. Let $\ker\omega = \{ X \in TM : i(X)\omega = 0 \}$

Then $C^\infty(\ker\omega) = \{$smooth vector fields V s.t. $\forall x \in M$, $V_x \in \ker\omega_x\}$.

forms induces a smooth (possibly singular) foliation $\mathcal{F}$ on M.

$$T\mathcal{F} \subseteq TM \to M$$

$$(\text{ie. } \Gamma(T\mathcal{F}) = C^\infty(\ker\omega).)$$

$$T_x\mathcal{F} = \operatorname{span}\left(C^\infty(\ker\omega)\big|_x\right).$$

Pf. Sp $X, Y \in \Gamma(\ker\omega)$ s.t.

$i(X)\omega = 0 = i(Y)\omega$ at each pt.

Then $i([X,Y])\omega =$

$$i(\mathcal{L}_X Y)\omega = [\mathcal{L}_X, i(Y)]\omega$$

$$= \mathcal{L}_X(i(Y)\omega) - i(Y)\mathcal{L}_X\omega$$

$$= i(Y)\left(d(i(X)\omega) + i(X)(d\omega)\right) = 0$$

$\therefore [X,Y] \in \Gamma(\ker\omega)$. By the Frobenius thm, $C^\infty(\ker\omega)$ is integrable. $\boxtimes$

<u>Lemma</u> If α is a closed 1-form on M
$(M,\mathcal{F})$ foliation of codim 1 s.t. $T\mathcal{F} = \ker\alpha$.
Then for a metric g on M, consider
$V_\alpha = \alpha^\#$ = vector field dual of α
(i.e. $\alpha(X) = \langle V_\alpha, X\rangle$.)
V_α is an infinitesimal automorphism of $\mathcal{F}$ $\Leftrightarrow$ $|\alpha|^2$ is basic.

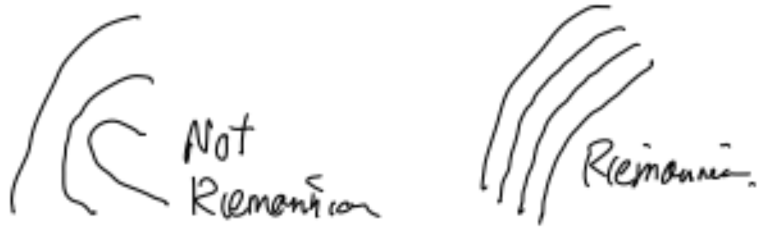
$\exists\, \phi_t : (M,\mathcal{F}) \to (M,\mathcal{F})$ family of foliated maps
$\phi_0 = Id$. — diffeomorphisms that map leaves to leaves.
s.t. $\frac{\partial}{\partial t}\phi_t = V_\alpha$.

a fcn $f \in C^\infty(M)$ is <u>basic</u> if it is constant on the leaves.

Lemma – (M,g). $\exists$ a harmonic 1-form of constant length $\Longleftrightarrow$ $\exists$ codim 1 minimal Riemannian foliation on (M,g).

leaves locally equidistant

mean curvature $= 0$ $\Longleftrightarrow$ leaves are minimal submanifolds.



Pf. With given α harmonic 1-form of const length

Rescale so $|\alpha| = 1$.

Then α is the transverse volume form for $\ker \alpha$. Then, by Rummler's formula

$$d\chi = -\kappa \wedge \alpha + \varphi_0$$

vol form of leaf

$= 0$

mean curvature form.

$0 \Longleftrightarrow$ perp space is involutive

$\Rightarrow K = 0$ and $\varphi_0 = 0$

$\Downarrow$

$\alpha^\#$ foliation is minimal (i.e.: totally geodesic)

$\Downarrow$

($\ker\alpha$ is Riemannian)

$\Downarrow$

$(\alpha^\#)^\perp$ is a foliation (knew that already).

Also $d(\underbrace{*\alpha}_{n-1 \text{ form}}) = 0$. $*\alpha$ = volume form of foliation ($\ker\alpha$)

$d(*\alpha) = 0 \Rightarrow$

$$= -\underset{\substack{\shortparallel \\ 0}}{K_1^\perp} *\alpha + \underset{\substack{\shortparallel \\ 0}}{\varphi_0^\perp}$$

$\therefore$ The foliation $\ker\alpha$ is also <u>minimal.</u>

$\therefore$ α harmonic form of length 1

$\therefore$ $\Rightarrow \ker\alpha$ is codim 1 Riem foliation that is minimal.

(Converse is also true.)

Lemma – If α & β are harmonic 1-forms of constant length on (M, g).
Then $\ker(\alpha \wedge \beta)$ is the tangent bundle to a Riem foliation $\Longleftrightarrow$ (α, β) is a basic fcn.

New proof of Torus metric result:
(only formal metrics on T^2 are flat metrics).

Pf. Suppose (T^2, g) is formal
$\Rightarrow \exists$ 1-form α that is harmonic & constant length.
$\Rightarrow *\alpha$ is also harmonic & constant length.

and $(\alpha, *\alpha)\, d\mathrm{vol} = \alpha \wedge *\alpha =$.

$\alpha \wedge **\alpha = \pm \alpha \wedge \alpha = 0$

$(\alpha, *\alpha) = 0$ (in fact α & $*\alpha$ are perpendicular)

$\ker \alpha$ & $\ker(*\alpha)$ give trivial Riem flows

$\Longleftrightarrow$ isometric flows.

$\Longleftrightarrow (T^2, g)$ is globally symmetric

$\Rightarrow$ curvature is constant

$\Rightarrow K = 0.$ $\blacksquare$ (by GB)