Using harmonic forms to learn about topology:

① $* : \Omega^k(M) \to \Omega^{n-k}(M)$.

If α is harmonic $\in \Omega^k$

$\Leftrightarrow \Delta\alpha = 0 \Leftrightarrow d\alpha = 0$ and $\delta\alpha = 0$

$\Leftrightarrow d\alpha = 0$ and $d*\alpha = 0$

and $*^2 = (-1)^{k(n-k)}$ on Ω^k.

$\Longrightarrow *\alpha$ is harmonic.

$\therefore \; * : H^k \to H^{n-k}$ isomorphically.

(Poincaré duality)

② $\dim(H^k(M)) < \infty$.

Example: What are the formal metrics on closed, oriented surfaces?

① Sphere ~ any metric is formal

② Surface of genus ≥ 2.

If α is 1-form, then α must have zeros (Hopf Index Thm $\Rightarrow \chi(M) < 0$, $\chi(M) =$ sum of indices of zeros of a nondegenerate v. field. $\Rightarrow$ dualizing – no 1-forms is nonvanishing.) $\Rightarrow$ every harmonic 1-form has zeros $\Rightarrow$ $\alpha \wedge \alpha$ has zeros.

If the metric were formal, this would have to be $C \cdot$(volume form) $\neq 0$ always.

$\therefore$ The metric can't be formal.

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③ Torus - what metrics are formal?

Interlude

Lemma - If (M,g) is formal, and α is a nontrivial harmonic form, then α has constant length.

$\ell(\alpha) = (\alpha,\alpha)^{1/2}$ ← pointwise inner product

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Pf: With above, α & $*\alpha$ are harmonic $\Rightarrow$ $\alpha \wedge *\alpha$ is harmonic

$$\underset{(\alpha,\alpha)(\mathrm{vol})}{\overset{\parallel}{\alpha \wedge *\alpha}} \leftarrow \text{for this to be harmonic, } (\alpha,\alpha) \text{ must be constant.}$$

Lemma More generally, if (M,g) is formal, if α, β are harmonic k-forms, then (α,β) is constant. (Same proof)

Suppose (T^2, g) is formal

Bochner Formula for 1-forms α:

$$\frac{1}{2}\Delta(|\alpha|^2) = (\Delta\alpha,\alpha) - |\nabla\alpha|^2 - \underbrace{\mathrm{Ric}}_{\text{Ricci Curvature}}(\alpha^\#, \alpha^\#)$$

Let α be a harmonic 1-form $\Rightarrow |\alpha|^2 = $ constant.

$$\Delta\alpha = 0$$

Bochner $\Rightarrow$ $$0 = -|\nabla\alpha|^2 - \underbrace{Ric(\alpha^\#, \alpha^\#)}_{\text{surfaces}}$$

$$\Rightarrow K = -|\nabla\alpha|^2 \leq 0 \qquad K = \text{Gauss curvature.}$$

Gauss Bonnet $\Rightarrow$ $$\underset{\overset{\shortparallel}{0}}{\chi(M)} = \frac{1}{2\pi}\int_M K\,dA \Rightarrow K = 0$$

$\Rightarrow g$ is flat.

Conversely, if g is flat, then g is formal.

$$\therefore (T^2, g) \text{ is formal} \Leftrightarrow g \text{ is flat.}$$

Smattering of known results about formal metrics:

Thms (Kotschick) (M^n, g) formal,

1. The real Betti #s of M satisfy $b_k(M) \leq b_k(T^n)$
2. If $n = 4m$, $b^{\pm}_{2m}(M) \leq b^{\pm}_{2m}(T^n)$
3. $b_1(M) \neq n-1$.

Thm (Kotschick): closed formal (M^n, g). If $b_1(M) = k$, $\exists$ smooth submersion $\pi : M \to T^k$, π^* is injection of cohomological algebras. If $b_1(M) = n$, $M \cong T^n$, and every formal metric is flat.

Thm - If (M,g) is a 3-manifold, then
$\exists$ formal metric on $M \Leftrightarrow M$ fibers over S^1.

Thm (M^n, g) with $n \leq 4$ has a formal metric
$\Leftrightarrow M$ has the real cohomology algebra of a compact symmetric space.

Thm Every (M^n, g) admits an open set of nonformal metric as long as M^n is not a rational homology sphere. (locally screw up Bochner formula)

Hopf Product Conj: $\nexists\, g$ on $S^2 \times S^2$ with $\sec(g) > 0$.

(2015) C. Bär ① If a 4-manifold is formal with $\sec(g) > 0$ then $M \cong S^4$ or $\mathbb{C}P^2$.

② If a metric $S^2 \times S^2$ has the property that a harmonic 2-form α has length that is not too nonconstant, then Hopf conj holds.
