Formality

(M, g) Riemannian manifold
(assume closed, oriented, connected)

Background:

$$d = d^k : \Omega^k(M) \to \Omega^{k+1}(M)$$

exterior derivative

{smooth k-forms on M}.

Leibniz Rule: If $\alpha \in \Omega^k$, β, Ω^r

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

$$d^2 = 0$$

de Rham cohomology

$$H^k(M) = \frac{\ker d^k}{\operatorname{Im} d^{k-1}}$$

$\leftarrow$ finite-dim vector space.

Map: $H^k(M) \times H^l(M) \to H^{k+l}(M)$

$$[\alpha] \wedge [\beta] \to [\alpha \wedge \beta].$$

cap product on cohomology

actually makes sense.

---

given metric g on M $\quad \langle \circ, \circ \rangle$
+def inner product on each T_pM

near $x \in M$, choose $\{e_1, \dots, e_n\}$ o-n basis for T_pM for p near x.

Dual basis $\{e_j\}$: e^j is a locally-defined 1-form s.t. $e^j(e_k) = \delta^j_k$ ← declare this basis of 1-forms to be orthonormal. ⇝ inner product on each T_p^*M.

⇝ extend to a basis of $\Lambda^k T_p^* M$. by declaring $\{e^{j_1} \wedge \dots \wedge e^{j_k}\}$ to be o-n.

⇝ gives inner product on each $\Lambda^k T_p^* M$.

⇝ label (α_p, β_p) = inner product of α_p, β_p in

$\ast$ operator:

$$\ast(e^1 \wedge e^2) = e^3 \wedge \dots \wedge e^n$$

$$\ast(e^1 \wedge e^2 \wedge e^3) = e^4 \wedge \dots \wedge e^n$$

$\ast$(basis element) = other basis element s.t.

$$(\text{basis elt}) \wedge \ast(\text{basis elt}) = \underbrace{e^1 \wedge \dots \wedge e^n}_{\text{agrees with volume form } d\mathrm{vol}}$$

→ extend to diff forms

$$\ast : \Omega^k(M) \to \Omega^{n-k}(M),$$

where $n = \dim M$.

Fact: If $\alpha, \beta \in \Omega^k$

$$\alpha \wedge \ast\beta = (\alpha, \beta)\, d\mathrm{vol}.$$

L^2 inner product on $\Omega^k(M)$

$$\langle \alpha, \beta \rangle := \int_M (\alpha, \overline{\beta})_p \, dvol .$$

Take closure of $\Omega^k(M)$ w.r.t. this inner product
completion

$\Rightarrow L^2(\Omega^k(M)) \leftarrow$ Hilbert space.

Δ The Laplacian is a differential operator defined on $\Omega^k(M)$.

$$\Delta : (d+\delta)^2 = d\delta + \delta d : \Omega^k(M) \circlearrowleft,$$

where $\delta = L^2$-adjoint of $d : \Omega^{k-1}(M) \to \Omega^k(M)$

$$\delta^k : \Omega^k(M) \to \Omega^{k-1}(M)$$

Cool formula: $\delta = (-1)^{nk+n+1} * d * : \Omega^k(M) \to \Omega^{k-1}(M)$

We call a differential form $\alpha \in \Omega^k(M)$ harmonic if $\Delta \alpha = 0 \Leftrightarrow d\alpha = 0$ and $\delta \alpha = 0$

If $\alpha \in \Omega^k$,

$$(d+\delta)^2 \alpha = (d+\delta)(d+\delta)\alpha$$

$$= \underset{0}{d^2\alpha} + \delta d \alpha + d\delta\alpha + \underset{0}{\delta^2 \alpha}$$

$$= \delta d \alpha + d \delta \alpha \in \Omega^k$$

Thm (de Rham) In each cohomology class $[\alpha]$, there exists a unique harmonic form, and this form is the form with minimum L^2-norm.

$(d\alpha = 0)$
$\alpha + d\tau \in [\alpha]$

Can we compute the cup product using harmonic forms?

If $\alpha \in \Omega^k$, $\Delta\alpha = 0$

$\beta \in \Omega^r$, $\Delta\beta = 0$

$$[\alpha \wedge \beta] \in H^{k+r}(M).$$

Question: Is $\alpha \wedge \beta$ harmonic?

Answer: Sometimes.

Defn: We say (M, g) is formal (g is formal) if the wedge product of any harmonic forms is harmonic.

Notes about the <u>formal</u> condition on a metric.

① Every compact (globally) symmetric space is <u>formal</u>.

(<u>Pf</u>: harmonic forms = invariant forms

clear: invariant $\wedge$ invariant is invariant.

QED)

② If M^n is a rational homology sphere.

(i.e. $H^0(M)$ and $H^n(M)$ are the only nonzero cohomology groups)

$\Rightarrow$ the only harmonic forms are constants & (const) dvol.

$\Rightarrow$ It's formal.

---

③ But... locally symmetric spaces need not be formal, formal manifolds need not be symmetric.
