1. What is the eta invariant?

The eta invariant was introduced in the famous paper of Atiyah, Patodi, and Singer (see [1]), in order to produce an index theorem for manifolds with boundary. The eta invariant of a linear self-adjoint operator is roughly the difference between the number of positive eigenvalues and the number of negative eigenvalues. One problem with this idea is that it does not make sense in infinite dimensions. However, we will have a way of **regularizing** to make this quantity well-defined for differential operators; this is similar to the zeta-function regularization of the determinant of the Laplacian and methods used by physicists to regularize quantities that are computed using divergent integrals. In fact, just as the zeta function of elliptic operators is analogous to the Riemann zeta function, the eta function is analogous to Dirichlet L-functions. Assume that we know the eigenvalues $\{\lambda\}$with multiplicity of a(n essentially) self-adjoint (usually first order) differential operator $D : C^{\infty}(E) \to C^{\infty}(E)$ on sections of a vector bundle $E \to M$, where M is a closed Riemannian manifold. We define the eta function $\eta(s)$ to be

$$\eta(s) = \sum_{\lambda} sgn(\lambda) |\lambda|^{-s},$$

where we define $sgn(0) = 0$. It turns out that the eta function is holomorphic in s for large $\mathrm{Re}(s)$, if D is elliptic; we will discuss this later. This is the zeta function if D has nonnegative eigenvalues. The **eta invariant** is

$$\eta(0),$$

which means that we analytically continue to $s = 0$. We see that this quantity is formally the number of positive eigenvalues minus the number of negative eigenvalues, $\infty - \infty$. Note that there is no reason to expect that this number is defined (ie $\eta(s)$ is regular at $s = 0$; it turns out that it often is) or that it is an integer (often it is not).

Remark: we can define these for pseudodifferential operators as well.

Boring example: Let $D = \frac{1}{i}\frac{d}{d\theta}$, a differential operator acting on complex-valued functions on the circle. (In the notation above, $M = S^1$, $E = M \times \mathbb{C}$.) We remark that this is the most elementary example of a Dirac operator. We will now compute the eta invariant of this operator. Observe that the eigensections of this operator are the functions $e^{in\theta}$ corresponding to eigenvalue $n \in \mathbb{Z}$, and in fact these eigenfunctions form an orthogonal basis of $L^2(S^1)$. Therefore, the eta function is

$$\begin{aligned}\eta(s) &= \sum_{\lambda} sgn(\lambda)|\lambda|^{-s} \\ &= \sum_{n\in\mathbb{Z}} sgn(n)|n|^{-s} = \sum_{n\in\mathbb{Z}_{>0}} n^{-s} - \sum_{n\in\mathbb{Z}_{>0}} n^{-s} = 0.\end{aligned}$$

Note that the sum above converges absolutely for Re$s > 1$, so the calculation is valid. Analytically continuing the function $\eta(s) = 0$ to $s = 0$, we obtain $\eta(0) = 0$.

Ostensibly less boring but just as boring example: Let $D = \frac{1}{i}\frac{d}{d\theta} + \frac{1}{2}$ on complex-valued functions on the circle. Then

$$\begin{aligned}\eta(s) &= \sum_{\lambda} sgn(\lambda)|\lambda|^{-s} \\ &= \sum_{n\in\mathbb{Z}} sgn\left(n+\frac{1}{2}\right)\left|n+\frac{1}{2}\right|^{-s} \\ &= \sum_{n\in\mathbb{Z}_{\geq 0}} \left(n+\frac{1}{2}\right)^{-s} - \sum_{n\in\mathbb{Z}_{\geq 0}} \left(n+\frac{1}{2}\right)^{-s} = 0,\end{aligned}$$

so again $\eta(0) = 0$.

Finally, a non-boring example: Let $D = \frac{1}{i}\frac{d}{d\theta} + c$ on complex-valued functions on the circle, where c is a real constant. Then

$$\begin{aligned}\eta(s) &= \sum_{\lambda} sgn(\lambda)|\lambda|^{-s} \\ &= \sum_{n\in\mathbb{Z}} sgn(n+c)|n+c|^{-s} = \sum_{n>-c} (n+c)^{-s} - \sum_{n<-c} (n+c)^{-s},\end{aligned}$$

which is nonzero in general. But now what do we do to obtain $\eta(0)$, or to obtain a closed-form expression for $\eta(s)$?

2. Families of Operators

Proposition 1. *Let Q_u be a C^∞ family of nonnegative self-adjoint operators with a complete system of eigenvalues λ_u for which the eigenfunctions form a basis for L^2, such that $\zeta(s)$ is defined and analytic at $s = 0$ and that there is a constant $N > 0$ such that $\zeta_u(s) = Tr(Q^{-s}) = \sum_{\lambda_u} \lambda_u^{-s}$ converges absolutely for $s > N$, and so that zero eigenspace depends differentiably on u (and thus is of constant dimension). Then the zeta function corresponding to Q_u satisfies*

$$\frac{d}{du}\zeta_u(s) = -s\ Tr\left(\dot{Q}_u Q_u^{-s-1}\right).$$

Proof. For a simple proof, if one may assume that the eigenvalues may be chosen to be differentiable in u (as in Rellich's Theorem), then one proves it like this:

$$\frac{d}{du}\zeta_u(s) = -s\sum \lambda_u^{-s-1}\dot{\lambda}_u = -s\ Tr\left(\dot{Q}Q^{-s-1}\right)$$

for large s, and then by the identity theorem, the analytic continuation satisfies the same equation. To prove it really using the ideas of Seeley, note that

$$\zeta_u(s) = \frac{1}{2\pi i}Tr\int_\Gamma \lambda^{-s}(Q_u - \lambda)^{-1}\,d\lambda$$

for some contour Γ enclosing the real axis. For large Res, convergence is guaranteed by estimates in [4]. Next we differentiate

$$(Q_u - \lambda)^{-1}(Q_u - \lambda) = I$$

to get

$$\frac{d}{du}(Q_u - \lambda)^{-1} = -(Q_u - \lambda)^{-1}\dot{Q}_u(Q_u - \lambda)^{-1}.$$

Then

$$\frac{d}{du}\zeta_u(s) = -\frac{1}{2\pi i}Tr\int_\Gamma \lambda^{-s}(Q_u - \lambda)^{-1}\dot{Q}_u(Q_u - \lambda)^{-1}\,d\lambda.$$

If it happened that Q_u is of very large order, $(Q_u - \lambda)^{-1}$ is trace class and has a continuous Schwarz kernel, and we may interchange trace with integral and commute operators within the trace. Then

$$\begin{aligned}
\frac{d}{du}\zeta_u(s) &= -\frac{1}{2\pi i}\int_\Gamma \lambda^{-s}Tr\left(\dot{Q}_u(Q_u - \lambda)^{-2}\right)d\lambda \\
&= .-\frac{1}{2\pi i}Tr\left(\dot{Q}_u\int_\Gamma \lambda^{-s}(Q_u - \lambda)^{-2}\,d\lambda\right) \\
&= -\frac{1}{2\pi i}Tr\left(\dot{Q}_u\int_\Gamma s\lambda^{-s-1}(Q_u - \lambda)^{-1}\,d\lambda\right)\ \text{(parts)} \\
&= -sTr\left(\dot{Q}_uQ_u^{-s-1}\right).
\end{aligned}$$

But what if Q_u is not of large order. Then if $m >> 0$ we still have Q_u^m satisfies the above, and

$$\begin{aligned}\frac{d}{du}\zeta_u(ms) &= -sTr\left(\frac{d}{du}(Q_u)^m Q_u^{-ms-m}\right)\\ &= -s\sum_j Tr\left(Q_u^j \dot{Q}_u Q_u^{m-j-1} Q_u^{-ms-m}\right)\\ &= -s\sum_j Tr\left(\dot{Q}_u Q_u^{-ms-1}\right)\\ &= -msTr\left(\dot{Q}_u Q_u^{-ms-1}\right),\end{aligned}$$

assuming one can commute the operators around in the trace, and thus the theorem is done. To carefully commute the operators in the trace, we see that if m is large enough

$$\begin{aligned}Tr\left(Q_u^j \dot{Q}_u Q_u^{m-j-1} Q_u^{-ms-m}\right) &= Tr\left(Q_u^j \dot{Q}_u Q_u^{-ms-j-1}\right)\\ &= Tr\left(Q_u^j \dot{Q}_u Q_u^{-ms/2} Q_u^{-ms/2-j-1}\right)\\ &= Tr\left(Q_u^{-ms/2-j-1} Q_u^j \dot{Q}_u Q_u^{-ms/2}\right)\\ &= Tr\left(Q_u^{-ms/2-1} \dot{Q}_u Q_u^{-ms/2}\right)\\ &= Tr\left(\dot{Q}_u Q_u^{-ms/2} Q_u^{-ms/2-1}\right)\\ &= Tr\left(\dot{Q}_u Q_u^{-ms-1}\right).\end{aligned}$$

□

Proposition 2. *Let D any self-adjoint operator for which $\eta(s)$ is defined and analytic at $s=0$, and that there is a constant $B>0$ such that $\sum_\lambda sgn(\lambda+c)|\lambda+c|^{-s}$ and $\sum_\lambda \left((\lambda+c)^2\right)^{-\frac{s+1}{2}}$ converge absolutely for $s>B$ and c in a certain interval such that $-c$ is not an eigenvalue of D for all c in that interval. Then the eta function $\eta_c(s)$ corresponding to the operator $D+c$ satisfies*

$$\frac{d}{dc}\eta_c(s) = -s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right),$$

where $\zeta_{(D+c)^2}$ is the zeta function corresponding to the nonnegative operator $(D+c)^2$, that is

$$\zeta_{(D+c)^2}(s) = \sum_{\mu>0}\mu^{-s},$$

where the sum is over all eigenvalues of the operator $(D+c)^2$. *In particular, if* D *is a first-order elliptic essentially self-adjoint differential operator, then* $\frac{d}{dc}\eta_c(0)$ *is the residue of the simple pole of the meromorphic function* $\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right)$ *at* $s=0$.

Remark: It is known that second-order essentially self-adjoint elliptic differential operators on a manifold of dimension n yield zeta functions with at most simple poles, and they are located at $s=\frac{n}{2}$, $s=\frac{n}{2}-1$, $s=\frac{n}{2}-2$, ... for n odd and at $s=\frac{n}{2}$, $s=\frac{n}{2}-1$, ... , $s=1$ for n even. Further, the residues at these poles are given by explicitly computable integrals of locally-defined functions.

Proof. We know that for each eigenvalue λ of D, $sgn\,(\lambda+c)$ does not vary with c in the interval. Then

$$\begin{aligned}
\eta_c(s) &= \sum_\lambda sgn\,(\lambda+c)\left((\lambda+c)^2\right)^{-s/2} \\
\frac{d}{dc}\eta_c(s) &= \sum_\lambda sgn\,(\lambda+c)\left(-\frac{s}{2}\left((\lambda+c)^2\right)^{-s/2-1}\right)2(\lambda+c) \\
&= -s\sum_\lambda sgn\,(\lambda+c)\,|\lambda+c|^{-s-2}(\lambda+c) \\
&= -s\sum_\lambda |\lambda+c|^{-s-1} \\
&= -s\sum_\lambda \left((\lambda+c)^2\right)^{-\frac{s+1}{2}} \\
&= -s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right).
\end{aligned}$$

Since both sides are analytic in s for large Res, the statement must remain true after analytic continuation. □

Proposition 3. *(More general version of the last proposition) For* c *in an open interval in* $\mathbb{R}$, *let* D_c *be a smooth family of self-adjoint operators for which* $\eta_c(s)=\eta_{D_c}(s)$ *is defined and analytic at* $s=0$ *for all* c, *and that there is a constant* $B>0$ *such that* $\sum_{\lambda_c} sgn\,(\lambda_c)\,|\lambda_c|^{-s}$ *and* $\sum_\lambda\left((\lambda_c)^2\right)^{-\frac{s+1}{2}}$ *converge absolutely for* $s>B$ *and* c *in the interval, such that* $\dim\ker D_c$ *is constant in* c. *Then the eta function* $\eta_c(s)$ *satisfies*

$$\frac{d}{dc}\eta_c(s) = -sTr\left(\dot{D}_c\left((D_c)^2\right)^{-\frac{s+1}{2}}\right).$$

In particular, if D_c is a family of first-order elliptic essentially self-adjoint differential operator, then $\frac{d}{dc}\eta_c(0)$ is the residue of the simple pole of the meromorphic function $Tr\left(\dot{D}_c\left((D_c)^2\right)^{-\frac{s+1}{2}}\right)$ at $s=0$.

Proof. We know that for each eigenvalue λ of D, $sgn(\lambda_c)$ does not vary with c in the interval. By the work of Rellich, we may assume that λ_c is differentiable in c. Then, for large Res,

$$\begin{aligned}
\eta_c(s) &= \sum_{\lambda_c} sgn(\lambda_c)\left((\lambda_c)^2\right)^{-s/2} \\
\frac{d}{dc}\eta_c(s) &= \sum_{\lambda_c} sgn(\lambda_c)\left(-\frac{s}{2}\left((\lambda_c)^2\right)^{-s/2-1}\right)2\lambda_c\frac{d}{dc}\lambda_c \\
&= -s\sum_{\lambda_c} sgn(\lambda_c)\left|\lambda_c\right|^{-s-2}\lambda_c\frac{d}{dc}\lambda_c \\
&= -s\sum_{\lambda_c\neq 0}\frac{\lambda_c}{|\lambda_c|}\left|\lambda_c\right|^{-s-2}\lambda_c\frac{d}{dc}\lambda_c \\
&= -s\sum_{\lambda_c\neq 0}\left|\lambda_c\right|^{1}\left|\lambda_c\right|^{-s-2}\frac{d}{dc}\lambda_c \\
&= -s\sum_{\lambda_c\neq 0}\left|\lambda_c\right|^{-s-1}\frac{d}{dc}\lambda_c \\
&= -s\sum_{\lambda}\left(\lambda_c^2\right)^{-\frac{s+1}{2}}\frac{d}{dc}\lambda_c \\
&= -sTr\left(\dot{D}_c\left((D_c)^2\right)^{-\frac{s+1}{2}}\right)
\end{aligned}$$

Since both sides are analytic in s, the statement must be true for all s. Again, this is ust the wimpy version of the proof; one needs to use the resolvent for a rigorous proof that does not require big hammers such as the Rellich theorem. □

Remark 4. *Until this moment we have always assumed that the variation does not change the quantity $sgn(\lambda_c)$. However, we note that one may exactly account for what happens to $\eta_c(s)$ as c varies in such a way that an eigenvalue goes through zero. That is, if λ' is the offending eigenvalue such that λ'_c passes through zero when $c=c_0$, we may instead consider the operator $D_c+\varepsilon P$, where P is the projection to the eigenspace corresponding to eigenvalue λ_c of D_c. It turns out that P can be written entirely in terms of powers of D_c and is thus a classical pseudodifferential operator as well. If ε is chosen to be sufficiently* *small to eliminate the difficulty at* c_0*. Further, observe that*

$$\eta_{D_c+\varepsilon P}(s) = \eta_c(s) + sgn(\lambda_c+\varepsilon)(\lambda_c+\varepsilon)^{-s} - sgn(\lambda_c)(\lambda_c)^{-s},$$

and upon analytic continuation we see that $\eta_{D_c+\varepsilon P}(0) = \eta_c(0) \pm 1$*. Thus, with no assumptions on passing through eigenvalues,* $\eta_c(0) \operatorname{mod} 1$ *is differentiable in* c*, and* $\eta_c(0)$ *may be calculated precisely by determining how many eigenvalues pass zero. The same holds for the zeta function.*

3. The Heat Kernel and Zeta Function

Now we collect some facts about the heat kernel and zeta functions.

Let L be a m^{th} order, nonnegative elliptic (classical pseudo-)differential operator on sections of a vector bundle E a closed Riemannian manifold M of dimension n whose principal symbol is the same as the $\frac{m}{2}$ power of the Laplacian, ie $\Delta^{m/2}$. Then the Cauchy problem for the heat equation has a unique solution among solutions that grow less than e^{t^2} in t:

$$\text{Problem:} \qquad \left(\frac{\partial}{\partial t} + L\right) u(x,t) = 0;\; u(x,0) = f(x)$$

$$\text{Solution:} \qquad u(x,t) = \int_M K(t,x,y) f(y)\, dV(y),$$

where $K(t,x,y) \in \operatorname{Hom}(E_y, E_x)$ is the **heat kernel** of L. The operator K satisfies the following asymptotic formula, for each $k \in \mathbb{Z}_{\geq 0}$, as $t \to 0$:

$$K(t,x,y) = \frac{1}{(4\pi t)^{n/m}} e^{-d(x,y)^2/4t} \left(\begin{array}{c} c_0(x,y) + c_1(x,y)\, t^{1/m} + \\ \ldots + c_k(x,y)\, t^{k/m} + \mathcal{O}\left(t^{(k+1)/m}\right) \end{array} \right),$$

where $d(x,y)$ is the Riemannian distance from x to y, and each c_j is smooth on $M \times M$, and $c_j(x,y) \in \operatorname{Hom}(E_y, E_x)$. In Eudlidean space, E is the trivial line bundle, $d(x,y) = |x-y|$, $c_0(x,y) = 1$, and $c_j(x,y) = 0$ for each $j > 0$. Plugging in $y = x$, we obtain, as $t \to 0$,

$$K(t,x,x) = \frac{1}{(4\pi t)^{n/m}} \left(\begin{array}{c} c_0(x,x) + c_1(x,x)\, t^{1/m} + \\ \ldots + c_k(x,x)\, t^{k/m} + \mathcal{O}\left(t^{(k+1)/m}\right) \end{array} \right).$$

In the differential case, the coefficients $c_j(x,y)$ can be calculated by directly plugging the asymptotic expansion into the differential equation and solving for them. They depend only on the metric and symbol of the operator along the minimal geodesic connecting x and y. Note that in order that $K(t,x,y)$ satisfies the initial condition, it must be true that $c_0(x,x) = 1$. Note that if e^{-tL} is the operator that maps $f(x)$ to $u(t,x)$, then

$$Tr\left(e^{-tL}\right) = \int_M Tr\left(K(t,x,x)\right)\, dV(x).$$

Under the additional assumption that L is an essentially self-adjoint (classical pseudo-)differential operator, then we may choose an orthonormal basis of $L^2(E)$ consisting of eigensections α_k of L corresponding to eigenvalues λ_k (counted with multiplicity), and we have

$$\begin{aligned}
K(t,x,y) &= \sum e^{-t\lambda_k}\alpha_k(x)\otimes\alpha_k(y)^*, \\
Tr\left(e^{-tL}\right) &= \int_M Tr\left(K(t,x,x)\right)\, dV(x) = \sum e^{-t\lambda_k}, \\
Tr\left(e^{-tL}\right) &= \frac{1}{(4\pi t)^{n/m}}\left(c_0 + c_1 t^{1/m} + ... + c_k t^k + \mathcal{O}\left(t^{k+1}\right)\right) \\
&\quad + \dim\ker(L)
\end{aligned}$$

and each sum absolutely and uniformly converges at each $t > 0$. Here, $c_j = \int_M Tr\left(c_j(x,x)\right)\, dV$.

Next, the zeta function of a nonnegative self-adjoint elliptic differential operator L is defined in analogy to the Riemann zeta function as

$$\zeta_L(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s}.$$

Note that in the case of the Laplacian $L = -\frac{d^2}{d\theta^2}$ on complex-valued functions on the circle, which has eigenvalues n^2 corresponding to orthogonal eigenfunctions $e^{\pm in\theta}$, we have

$$\zeta_L(s) = \sum_{n>0} 2n^{-2s} = 2\zeta^R(2s),$$

where $\zeta^R(s)$ is the Riemann zeta function. Note that

$$\lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t\lambda}dt,$$

so we have that

$$
\begin{aligned}
\zeta_L(s) &= \sum_{\lambda_k\neq 0}\lambda_k^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\sum_{\lambda_k\neq 0}e^{-t\lambda_k}\right)dt\\
&= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\int_M Tr\left(K\left(t,x,x\right)\right)\ dV\left(x\right)-\dim\ker L\right)dt\\
&= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\int_M Tr\left(K\left(t,x,x\right)\right)\ dV\left(x\right)-\dim\ker L\right)dt\\
&\quad +\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(\int_M Tr\left(K\left(t,x,x\right)\right)\ dV\left(x\right)-\dim\ker L\right)dt\\
&= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\frac{1}{(4\pi t)^{n/m}}\left(c_0+c_1t^{1/m}+...+c_Nt^{N/m}\ \right)\right)dt\\
&\quad +\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\sum e^{-t\lambda_k}-\dim\ker L\right.\\
&\quad \left.-\frac{1}{(4\pi t)^{n/m}}\left(c_0+c_1t^{1/m}+...+c_Nt^{N/m}\ \right)\right)dt\\
&\quad +\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(\sum e^{-t\lambda_k}-\dim\ker L\right)dt\\
&= \frac{1}{(4\pi)^{n/m}\Gamma(s)}\sum_{j=0}^N c_j\int_0^1 t^{s-1-\frac{n}{m}+\frac{j}{m}}dt+\phi_N(s)\\
&= \frac{1}{(4\pi)^{n/m}\Gamma(s)}\sum_{j=0}^N\frac{c_j}{s-\frac{n}{m}+\frac{j}{m}}+\phi_N(s)
\end{aligned}
$$

for large s, and this formula gives the meromorphic continuation of the zeta function $\zeta_L(s)$ (with $\phi_N(s)$ holomorphic for $\mathrm{Re}s>\frac{n}{2}-N$). Observe that, as stated earlier, in the differential case, $\zeta_L(s)$ has at most simple poles, and they are located at $s=\frac{n}{2}$, $s=\frac{n}{2}-1$, $s=\frac{n}{2}-2$, ... for n odd and at $s=\frac{n}{2}$, $s=\frac{n}{2}-1$, ... , $s=1$ for n even (note that $\frac{1}{\Gamma(s)}$ has a simple zero at each nonpositive integer). The residue of the pole at $s=\frac{n}{m}-\frac{j}{m}$ is $\frac{c_j}{(4\pi)^{n/m}\Gamma\left(\frac{n}{m}-\frac{j}{m}\right)}$.

We remark that many pseudodifferential operators have the same properties regarding the analytic continuation. For instance, if A is a self-adjoint differential operator and p is any positive real number, then $|A|^p = (A^2)^{p/2}$ is a pseudodifferential operator, and

$$\zeta_{|A|^p}(s) = \sum_{\lambda\neq 0} |\lambda|^{-ps} = \sum_{\lambda\neq 0} \left(\lambda^2\right)^{-ps/2} = \zeta_{A^2}\left(\frac{ps}{2}\right),$$

and its analytic continuation and poles can be obtained from those of ζ_{A^2}. In particular, $\zeta_{|A|^p}(0) = \zeta_{A^2}(0)$, and $\zeta'_{|A|^p}(0) = \frac{p}{2}\zeta'_{A^2}(0)$. In general, the asymptotic expansions of heat kernels corresponding to pseudodifferential operators may have powers of t that increment by $\frac{1}{2}$, and in addition logarithmic terms may appear. The logarithmic terms cause the corresponding zeta functions to have poles of higher order. According Seeley's paper [4], for any classical pseudodifferential operator A on a closed manifold M, the the restriction of the kernel of A^{-s} to the diagonal in $M \times M$ is meromorphic with poles only at $s = \frac{n-k}{m}$, $k = 0, 1, 2, ...$ (where m is the order of the operator, n is the dimension of M), and the pole $s = \frac{k-n}{m}$, and its residue is given by an explicit formula. The residues at $s = 0, -1, -2, ...$ vanish, and the value of the kernel at $s = 0$ is again given by an explicit formula.

Explicitly, note that $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1. From this we see from the formula above that

$$\zeta_L(0) = \frac{c_n}{(4\pi)^{n/m}},$$

and

$$c_{n/m} = \int c_{n/m}(x,x)$$

is explicitly calculable from the metric and the local symbol of the operator, in the differential case.

Now we may **return to the Nonboring Example:** Now we apply the Proposition to the operator $D + c = \frac{1}{i}\frac{d}{d\theta} + c$ on the circle. By the first proposition, we have that

$$\frac{d}{dc}\eta_c(s) = -s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right),$$

so that $\frac{d}{dc}\eta_c(0)$ is -2 times the residue of $\zeta_{(D+c)^2}(z)$ at $z = \frac{1}{2}$. But note that $(D+c)^2$ has the same principal symbol as the Laplacian, and thus its heat kernel satisfies

$$\begin{aligned} Tr\left(e^{-t(D+c)^2}\right) &= \frac{1}{\sqrt{4\pi t}}\begin{pmatrix} \int_0^{2\pi} c_0(x,x)\ d\theta + t\int_0^{2\pi} c_1(x,x)\ d\theta + \\ ... + t^N \int_0^{2\pi} c_N(x,x)\ d\theta + \mathcal{O}\left(t^{N+1}\right) \end{pmatrix} \\ &= \frac{1}{\sqrt{4\pi t}}\left(2\pi + tc_1 + ... + t^N c_N + \mathcal{O}\left(t^{N+1}\right)\right). \end{aligned}$$

Then

$$\begin{aligned}
\zeta_{(D+c)^2}(s) &= \frac{1}{(4\pi)^{1/2}\Gamma(s)}\frac{2\pi}{\left(s-\frac{1}{2}\right)} \\
&\quad + \frac{1}{(4\pi)^{1/2}\Gamma(s)}\sum_{j=1}^{N}\frac{c_j}{s-\frac{1}{2}+j} + \text{holomorphic}(s),
\end{aligned}$$

so the residue is $\frac{2\pi}{2\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)} = 1$. Thus, near $s=0$

$$\begin{aligned}
-s\zeta_{(D+c)^2}\left(\frac{s+1}{2}\right) &= -\frac{s}{\frac{s+1}{2}-\frac{1}{2}} = -2, \text{ so} \\
\frac{d}{dc}\eta_c(0) &= -2.
\end{aligned}$$

Since when $c=\frac{1}{2}$, $\eta_c(0)=0$, we have that

$$\eta_c(0) = -2\left(c-\frac{1}{2}\right) = 1-2c$$

for $0<c<1$. Note that the spectrum is invariant as $c\mapsto c+1$, so in fact

$$\eta_c(0) = -2\left(c-\frac{1}{2}\right) = (1-2c)\,\mathrm{mod}2\mathbb{Z},\ c\in\mathbb{R}\setminus\mathbb{Z}$$

We have seen that

$$\eta_c(0)=0,\ c\in\mathbb{Z}.$$

4. Relationship between zeta and eta

According to Seeley's famous paper [1], complex powers of pseudodifferential operators are again pseudodifferential. Thus, if A is a first order self-adjoint elliptic pseudo-differential operator, then

$$B_1 := \frac{3}{2}|A| + \frac{1}{2}A,\, B_2 := \frac{3}{2}|A| - \frac{1}{2}A$$

are also elliptic and pseudodifferential but are nonnegative. Let $\zeta_j(s)$ be the zeta function corresponding to B_j for $j=1,2$. Then if λ ranges over eigenvalues of A,

$$\begin{aligned}
\zeta_1(s)-\zeta_2(s) &= \sum_{\lambda\neq 0}\left(\frac{3}{2}|\lambda|+\frac{1}{2}\lambda\right)^{-s} - \sum_{\lambda\neq 0}\left(\frac{3}{2}|\lambda|-\frac{1}{2}\lambda\right)^{-s} \\
&= 2^{-s}\sum_{\lambda>0}\lambda^{-s} + \sum_{\lambda<0}|\lambda|^{-s} - \sum_{\lambda>0}\lambda^{-s} - 2^{-s}\sum_{\lambda<0}|\lambda|^{-s} \\
&= \left(2^{-s}-1\right)\sum_{\lambda\neq 0} sgn(\lambda)|\lambda|^{-s} = \left(2^{-s}-1\right)\eta_A(s).
\end{aligned}$$

Thus,

$$\eta_A(s) = \frac{\zeta_1(s) - \zeta_2(s)}{2^{-s} - 1}.$$

This gives the meromorphic continuation of the eta function. If each ζ_j has only simple poles and is regular at $s = 0$ (as it is for powers of self-adjoint elliptic differential operators), then $\eta_A(s)$ has only simple poles (including possibly at $s = 0$). The residue of $\eta_A(s)$ at $s = 0$ is

$$R(A) = -\frac{1}{\log 2}(\zeta_1(0) - \zeta_2(0)).$$

We need to show that $R(A)$ is in fact zero, and as a consequence we will be able to deduce that $\eta_A(s)$ is regular at $s = 0$. Note that by the formula for $\zeta_L(0)$ above, $R(A)$ is an integral of a locally determined quantity on the manifold.

5. Regularity of $\eta(s)$ at $s = 0$

The next step is to show that $R(A)$ is constant on a family of operators A_u. In order to allow for discontinuities produced by zero eigenvalues, we can write $\eta_u(s) = \eta'_u(s) + \eta''_u(s)$ as a sum of two parts, $\eta'_u(s)$ corresponding to the eigenvalues λ such that $|\lambda| < C$ and $\eta''_u(s)$ corresponding to eigenvalues λ such that $|\lambda| > C$, where C is interior to a spectral gap for $|A|$. The function $\eta'_u(s)$ is a finite sum of exponential function and is thus entire and also differentiable in u. Thus, if we let $\overline{\eta}_u(s)$ denote $\eta_u(s) \bmod \mathbb{Z}$, we see that $\overline{\eta}_u(0) = \overline{\eta}''_u(0)$. Thus, we may set assume change all eigenvalues λ of A_u such that $|\lambda| < C$ to 1 without changing $\overline{\eta}_u(0)$; this means that we may assume A_u is invertible for all u.

To prove $R(A)$ is constant on such a family of operators A_u, it suffices to show that for such a family A_u, $\dot{\eta}_0(0) = 0$. Let $B_u = |A_0| + u\dot{A}_0$, which is elliptic and positive for small u. Then by the propositions above,

$$\begin{aligned} \frac{d}{du}\eta_{A_u}(s) &= -sTr\left(\dot{A}_u\left((A_u)^2\right)^{-\frac{s+1}{2}}\right) = -sTr\left(\dot{A}_u B_u^{-s-1}\right), \\ \frac{d}{du}\zeta_{B_u}(s) &= -s\ Tr\left(\dot{B}_u B_u^{-s-1}\right) = -s\ Tr\left(\dot{A}_0 B_u^{-s-1}\right) \end{aligned}$$

so that these derivatives coincide at $u = 0$. By the above, $\frac{d}{du}\zeta_{B_u}(s)$ has a meromorphic continuation which is regular at $s = 0$, so the same must be true for $\frac{d}{du}\eta_{A_u}(s)$. By the above relationship between ζ and η, and the meromorphic continuation formula for $\zeta_L(s)$, we have (with

ζ_1, ζ_2 corresponding to $\frac{3}{2}|A_u| + \frac{1}{2}A_u, \frac{3}{2}|A_u| - \frac{1}{2}A_u$) that $\eta_{A_u}(s)$ is of the form

$$\begin{aligned} \eta_{A_u}(s) &= \frac{\zeta_1(s) - \zeta_2(s)}{2^{-s} - 1} \\ &= \frac{R(A_u)}{s} + \sum_{j=-n,\neq 0}^{k} \frac{a_j(u)}{s - \frac{j}{m}} + \phi_k(s,u), \end{aligned}$$

where $\phi_k(s,u)$ is a smooth map into the space of holomorphic functions on $\mathrm{Re}s > -\frac{k}{m}$. But then the residue of $\left.\frac{d}{du}\right|_{u=0} \eta_{A_u}(s)$ at $s = 0$ is then $\frac{d}{du}R(A_u)$, which must be zero by the latest calculations. Thus, $R(A_u)$ is constant in u. Thus $R(A)$ is a homotopy invariant of A.

By the comments at the beginning of this section, if $\overline{\eta}_{A_u}(s)$ is the eta function reduced modulo $\mathbb{Z}$, then $\left.\frac{d}{du}\right|_{u=0} \overline{\eta}_{A_u}(s)$ is holomorphic at $s = 0$, and its value there is given by an explicit integral formula constructed out of the complete symbols of A_0 and $\dot{A}_0$.

5.1. **Aside: a homotopy invariant for operators twisted by flat bundles.** A consequence of the above for flat bundles is as follows. Let $\alpha : \pi_1(M) \to U(N)$ be a unitary representation, and this defines a flat vector bundle $\widetilde{M} \times_a V_\alpha$ over M with Hermitian metric. If $A : C^\infty(M,E) \to C^\infty(M,E)$ is a differential operator acting on sections of E, then A extends naturally to $A_\alpha : C^\infty(M, E \otimes V_\alpha) \to C^\infty(M, E \otimes V_\alpha)$. Moreover, if A is self-adjoint, then A_α is also self-adjoint. Let

$$\widetilde{\eta}_\alpha(s,A) := \overline{\eta}_{A_\alpha}(s) - N\overline{\eta}_A(s).$$

Since the operators A_α and $A_N = A \oplus ... \oplus A$ (N times) are locally isomorphic, any invariant given by a local integral formula will coincide for the two operators. Thus, $R(A_\alpha) = R(A_N) = NR(A)$, so that $\widetilde{\eta}_\alpha(s,A)$ is regular at $s = 0$. By the above, $\left.\frac{d}{du}\right|_{u=0} \widetilde{\eta}_\alpha(s, A_u)$ is zero at $s = 0$, so that $\widetilde{\eta}_\alpha(s,A)$ is a homotopy invariant of A. If A is instead pseudodifferential, there is no unique way of defining A_α. However, using a partition of unity, we can construct an operator A_α whose complete symbol is $\sigma(A) \otimes \mathbf{1}_\alpha$. We have shown

Proposition 5. *(Notation as above) $\widetilde{\eta}_\alpha(0,A)$ is a finite homotopy invariant of A and takes values in $\mathbb{R}/\mathbb{Z}$.*

5.2. **Back to the regularity of the eta function. Reduced eta invariant:** If you replace η by $\xi = \frac{\eta + h}{2}$, where h is the dimension of the nullspace, all of the results above apply.

K-theory and self-adjoint symbols: Since $R(A)$ is a homotopy invariant of A (and with adjustment is a actually a stable homotopy invariant of the symbol $\sigma(A)$), it suffices to check that $R(A)=0$ for a sufficiently rich set of symbols that generate all K-theory classes. You can either use the Dirac operator or the boundary part of the signature operator on odd dimensional manifolds. Then, by invariance theory, the local integrand must be a Pontryagin-Chern form, and thus of even degree. Then, we have that $R(A)=0$ on odd-dimensional manifolds.

Theorem 6. *If M is an odd-dimensional manifold, and A is a self-adjoint elliptic pseudodifferential operator of positive order on M, then $\eta_A(s)$ is holomorphic at $s=0$.*

The even-dimensional case is much trickier, and you can see the proof in [3].

6. Another meromorphic continuation of the eta function

If A is an self-adjoint elliptic classical pseudodifferential operator of order d on a manifold of dimension n, observe that

$$\begin{aligned}
\eta_A(s) &= \sum_{\lambda} sgn(\lambda)\,|\lambda|^{-s} = \sum_{\lambda\neq 0} \lambda\,|\lambda|^{-s-1} = \sum_{\lambda\neq 0} \lambda\left(\lambda^2\right)^{-\frac{s+1}{2}} \\
&= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}-1}\left(\sum_{\lambda} \lambda e^{-t\lambda^2}\right) dt \\
&= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}-1} Tr\left(Ae^{-tA^2}\right) dt
\end{aligned}$$

Now, it turns out that $Tr\left(Ae^{-tA^2}\right)$ has an asymptotic expansion in powers of t, beginning with $t^{\frac{-n-d}{2d}}$, and so the integral gives an analytic expression for $\eta_A(s)$ for $\frac{s+1}{2} > \frac{-n-d}{2d}$, i.e. $s > \frac{-n-2d}{d} = -\frac{n}{d} - 2$. If A is a differential operator, then we have

$$Tr\left(Ae^{-t^2A^2}\right) = \sum_{k=0}^{N} c_k(A)\, t^{\frac{k-n-d}{2d}} + \mathcal{O}\left(t^{\frac{N-n-d+1}{2d}}\right),$$

where as in the heat asymptotic expansion, if A is differential, $c_k(A) = \int c_k(A,x)\ dV(x)$, where $c_k(A,x)$ is a locally determined quantity.

Thus, the meromorphic continuation of $\eta_A(s)$ (for differential operators and classical pseudodifferential operators) is given by

$$\begin{aligned}
\eta_A(s) &= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^1 t^{\frac{s+1}{2}-1}\left(\sum_{k=0}^{N} c_k(A)\, t^{\frac{k-n-d}{2d}}\right)dt \\
&\quad +\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^1 t^{\frac{s+1}{2}-1}\left(Tr\left(Ae^{-tA^2}\right)-\sum_{k=0}^{N} c_k(A)\, t^{\frac{k-n-d}{2d}}\right)dt \\
&\quad +\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_1^\infty t^{\frac{s+1}{2}-1} Tr\left(Ae^{-tA^2}\right)dt \\
&= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\sum_{k=0}^{N} c_k(A)\int_0^1 t^{\frac{1}{2}s+\frac{1}{2d}(k-d-n)-\frac{1}{2}}dt + \text{holomorphic}(s) \\
&= \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\sum_{k=0}^{N} \frac{c_k(A)}{\frac{1}{2}s+\frac{1}{2d}(k-d-n)+\frac{1}{2}} + \text{holomorphic}(s)
\end{aligned}$$

This formula shows that the residue at $s = 0$ occurs when

$$\begin{aligned}
0 &= \frac{1}{2d}(k-d-n)+\frac{1}{2}, \text{ or} \\
k &= 2d\left(-\frac{1}{2d}(-d-n)-\frac{1}{2}\right) = n,
\end{aligned}$$

or

$$res_{s=0}(\eta_A(s)) = \frac{2c_n(A)}{\Gamma\left(\frac{1}{2}\right)}.$$

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