Unitary Equivalence of Normal Matrices over Topological Spaces

GAGA Seminar

Fall 2022

Theorem (Spectral Theorem)

Every normal element of $\mathrm{M}(n, \mathbb{C})$ *is diagonalizable; i.e., is unitarily equivalent to a diagonal matrix.*

Question

If X is a topological space, is every normal element in $\mathrm{M}(n, C(X))$ diagonalizable? In other words, if $A \in \mathrm{M}(n, C(X))$ is normal, does there exist a unitary $U \in \mathrm{M}(n, C(X))$ such that U^*AU is a diagonal matrix?

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No in general – R. Kadison gave a counterexample in $\mathrm{M}(2, C(S^4))$.

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Example 1: $A \in \mathrm{M}(2, C[-1,1])$

$$A(x) = \begin{cases} \begin{pmatrix} x & x \\ x & x \end{pmatrix} & x \geq 0 \\ \\ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} & x < 0. \end{cases}$$

If U^*AU diagonal, then

$$U(x) = \begin{pmatrix} f(x) & g(x) \\ -f(x) & g(x) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} g(x) & f(x) \\ g(x) & -f(x) \end{pmatrix}, \quad x > 0$$

$$|f(x)| = |g(x)| = \frac{1}{\sqrt{2}}$$

$$U(x) = \begin{pmatrix} h(x) & 0 \\ 0 & k(x) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & h(x) \\ k(x) & 0 \end{pmatrix}, \quad x < 0$$

$$|h(x)| = |k(x)| = 1.$$

A in $\mathrm{M}(n, C(X))$ is *multiplicity-free* if $A(x)$ has distinct eigenvalues for each x in X.

Equivalently, A is multiplicity-free if its characteristic polynomial $p(x, \lambda) = \det(\lambda I - A(x))$ has n distinct zeros for each x in X.

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The functions $d_i : X \longrightarrow \mathbb{C}$, $1 \leq i \leq n$ are continuous and thus the characteristic polynomial of A globally splits:

$$p(x, \lambda) = \prod_{i=1}^{n} (\lambda - d_i(x)).$$

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$$A(z) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}.$$

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Problem: the zeros of the characteristic polynomial exhibit *nontrivial monodromy*.

Example 3: $A \in \mathrm{M}(2, C(S^2))$

$$A(x,y,z) = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix}.$$

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However, the eigenspaces of $A(x, y, z)$ associated to the eigenvalue 1 define a nontrivial complex line bundle E_1 over S^2, whence E_1 does not admit a global nonvanishing section. This implies that A cannot be diagonalized.

Theorem (Grove and Pedersen, 1984)

Let X be a 2-connected ($\pi_1(X) = \pi_2(X) = 0$) compact CW complex and suppose that $A \in \mathrm{M}(n, C(X))$ is normal and multiplicity-free. Then A is diagonalizable.

Proof:

$\pi_1(X) = 0$ implies that the characteristic polynomial of A globally splits.

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$\pi_1(X) = 0$ and $\pi_2(X) = 0$ imply that $H^2(X; \mathbb{Z}) = 0$, which in turn implies that $E_1, \ldots, E_n$ admit globally nonvanishing sections $d_1, \ldots, d_n$.

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Apply Gram-Schmidt to $d_1(x), \ldots, d_n(x)$ for each x to obtain vectors $e_1(x), \ldots, e_n(x)$; these form the columns of a unitary matrix that diagonalizes A.

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However, this is not sufficient in general: see Example 3.

Theorem (Friedman-Park, 2014)

Let X be a connected CW complex and suppose A, B in $\mathrm{M}(n, C(X))$ are normal, multiplicity-free, and have the same characteristic polynomial. Then there exists a cohomology class $[\theta(A,B)]$ in $H^2(X, \text{“}\mathbb{Z}^n\text{”})$ with the property that A, B are unitarily equivalent if and only if $[\theta(A,B)] = 0$.

Corollary

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Proofs: In both cases, we have $H^2(X, \text{``}\mathbb{Z}^n\text{''}) = 0$.

Corollary

Suppose X is a CW-complex that contains a countable number of 2-cells. Then the number of unitary equivalence classes of multiplicity-free normal matrices over $C(X)$ with a given characteristic polynomial is countable.

Proposition

Suppose A and B in $\mathrm{M}(n, C(X))$ are normal, multiplicity free, and have a common characteristic polynomial with trivial monodromy. Continuously order the eigenvalues $\lambda_1(x), \ldots, \lambda_n(x)$ of $A(x)$ and $B(x)$ and let $E_1, \ldots, E_n$ and $F_1, \ldots, F_n$ be the corresponding eigenbundles of A and B respectively. Then

$$[\theta(A, B)] = \bigoplus_{i=1}^{n} c^1(\mathrm{Hom}(E_i, F_i)).$$

Corollary

Suppose that $A \in \mathrm{M}(n, C(X))$ *is normal and multiplicity-free and that the characteristic polynomial of* A *splits over* $C(X)$. *Choose an ordering* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *for the eigenvalues of* A, *and let* D *be the diagonal matrix with diagonal entries* $\lambda_1, \lambda_2, \ldots, \lambda_n$. *Then*

$$\theta(D, A) = c_1(V_1) \oplus c_1(V_2) \oplus \cdots \oplus c_1(V_n),$$

where $V_1, V_2, \ldots, V_n$ *are the eigenbundles corresponding to the eigenvalues of* A. *Thus* A *is diagonalizable if and only if* $V_1, V_2, \ldots, V_n$ *all have trivial first Chern class.*

Lemma

For $k > 0$, the elementary symmetric polynomials s_k evaluated at $c_1(V_1), c_1(V_2), \ldots, c_1(V_n)$ vanish.

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Proof.

For each i, let $c(V_i)$ denote the total Chern class of V_i. Because each V_i is a line bundle, we have that $c(V_i) = 1 + c_1(V_i)$. By the Whitney product formula,

$$1 = c(\Theta^n(X)) = c\left(\oplus_{i=1}^n V_i\right) = \prod_{i=1}^n c(V_i)$$

$$= \prod_{i=1}^n (1 + c_1(V_i)) = 1 + \sum_{k=1}^n s_k\left(c_1(V_1), c_1(V_2), \ldots c_1(V_n)\right).$$

□

Proposition

Suppose that A in $\mathrm{M}(n, C(\mathbb{C}P^m))$ *is normal and multiplicity-free and that* $m > 1$. *Then A is diagonalizable.*

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Proof.

$$(s_1(x_1, x_2, \ldots, x_n))^2 - 2s_2(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots x_n^2$$

and so

$$(c_1(V_1))^2 + (c_1(V_2))^2 + \cdots + (c_1(V_n))^2 = 0.$$

□

Proof.

$$H^*(\mathbb{C}P^m) \cong \mathbb{Z}[\alpha]/\alpha^{m+1} \Rightarrow c_1(V_i) = k_i\alpha,\ k_i \in \mathbb{Z}$$

$$0 = \sum_{i=1}^{n} (c_1(V_i))^2 = \sum_{i=1}^{n} (k_i\alpha)^2 = \left(\sum_{i=1}^{n} k_i^2\right) \alpha^2 \in H^4(\mathbb{C}P^m).$$

Because $m > 1$, the class α^2 is a generator of $H^4(\mathbb{C}P^m) \cong \mathbb{Z}$, and therefore all the integers k_i are zero. Therefore $c_1(V_i) = 0$ for all $1 \leq i \leq n$. □

Proposition

Suppose that X is a CW complex and let $\mu, \widetilde{\mu} \in C(X)[\lambda]$ be multiplicity-free polynomials that split over $C(X)$ and have the same degree. Then the number of unitary equivalence classes of normal matrices over X with characteristic polynomial μ is equal to the number of unitary equivalence class of normal matrices over X with characteristic polynomial $\widetilde{\mu}$.

Theorem

Let X be a CW complex with $\dim(X) \leq 3$, and let $\mu \in C(X)[\lambda]$ be a multiplicity free polynomial of degree n that splits over $C(X)$. There is a bijection between the set of unitary equivalence classes of $n \times n$ normal matrices with characteristic polynomial μ and elements of the group $(H^2(X))^{n-1} = \oplus_{i=1}^{n-1} H^2(X)$.

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Our $\mathbb{C}P^m$ example shows that the hypothesis $\dim(X) \leq 3$ is necessary.

Chern-Weil Theory

V is a smooth complex vector bundle over a smooth manifold X

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Let dP denote the matrix of one–forms obtained by applying the exterior derivative d to each entry of P. Then $\frac{1}{2\pi i}\,\mathrm{tr}(PdPdP)$ is a closed two-form whose class $H^2_{deR}(X)$ in the de Rham cohomology of X is $c_1(V)$.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M(2, C(S^2))$$

$$\begin{aligned} a_{11} &= x^2 + x^3 + y^2 + xy^2 + i(1-x)z^2 \\ a_{12} &= (y + iz)(x^2 + y^2 - iz^2) \\ a_{21} &= (y - iz)(x^2 + y^2 - iz^2) \\ a_{22} &= x^2 - x^3 + y^2 - xy^2 + i(1+x)z^2 \end{aligned}$$

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\end{aligned}$$

$$\begin{aligned}
\mu_A(\lambda) &= \lambda^2 - 2(x^2 + y^2 + iz^2)\lambda + 4i(x^2 + y^2)z^2 \\
&= \Big(\lambda - 2(x^2 + y^2)\Big)\Big(\lambda - 2iz^2\Big).
\end{aligned}$$

Spectral projection associated to $2(x^2 + y^2)$ is

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Switch to polar coordinates:

$$P = \frac{1}{2}\begin{pmatrix} 1+\sin\phi\cos\theta & \sin\phi\sin\theta + i\cos\phi \\ \sin\phi\sin\theta - i\cos\phi & 1-\sin\phi\cos\theta \end{pmatrix}$$

$$\mathrm{tr}(PdPdP) = \frac{i}{2} \sin\phi \, d\theta d\phi$$

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$$\frac{1}{2\pi i} \int_{S^2} \frac{i}{2} \sin\phi \, d\theta d\phi = 1 \neq 0,$$

so A is not diagonalizable.

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$$R := \begin{pmatrix} q_{11}p_{11} & \cdots & q_{11}p_{n1} & \cdots & q_{1n}p_{11} & \cdots & q_{1n}p_{n1} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ q_{11}p_{1n} & \cdots & q_{11}p_{nn} & \cdots & q_{1n}p_{1n} & \cdots & q_{1n}p_{nn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ q_{n1}p_{11} & \cdots & q_{n1}p_{n1} & \cdots & q_{1n}p_{11} & \cdots & q_{1n}p_{n1} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ q_{n1}p_{1n} & \cdots & q_{n1}p_{nn} & \cdots & q_{nn}p_{1n} & \cdots & q_{nn}p_{nn} \end{pmatrix}$$

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$$c_1(\mathrm{Hom}(V, W)) = \left[\frac{1}{2\pi i} \mathrm{tr}(RdR\, dR) \right]$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M(2, C(S^2))$$

$$b_{11} = x^2 - x^2 z + y^2 - y^2 z + iz^2(z+1)$$
$$b_{12} = (x+iy)(ix^2 + iy^2 + z^2)$$
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$$\mu_B(\lambda) = \mu_A(\lambda) = \Big(\lambda - 2(x^2+y^2)\Big)\Big(\lambda - 2iz^2\Big)$$

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R=

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$$\mathrm{tr}(RdRdR) = i(zdxdy - ydxdz + xdydz) = -i \sin\phi d\theta d\phi$$

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Thus

$$\int_{S^2} \frac{1}{2\pi i}\,\mathrm{tr}(R_1\, dR_1\, dR_1) = \frac{1}{2\pi i}\int_0^{\pi}\int_0^{2\pi} -i\sin\phi\, d\theta d\phi = -2 \neq 0,$$

and therefore A and B are not unitarily equivalent.