Traces and Determinants

Let A be an $n \times n$ matrix with complex entries:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Then

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$$

and

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Properties of trace: For A and B in $\mathrm{M}(n, \mathbb{C})$ and S in $\mathrm{GL}(n, \mathbb{C})$,

- $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$;
- $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
- $\operatorname{tr}(SAS^{-1}) = \operatorname{tr} A$;
- The trace of A is the sum of the eigenvalues of A.

Properties of determinant:

- $\det(AB) = \det(BA) = (\det A)(\det B)$;
- $\det(SAS^{-1}) = \det A$;
- The determinant of A is the product of the eigenvalues of A.

Define the *exponential* of A as

$$\exp A := \sum_{k=0}^{\infty} \frac{1}{k\,!} A^k.$$

Warning: In general $\exp(A + B) \neq (\exp A)(\exp B)$ unless A and B commute.

Theorem: $\det(\exp A) = e^{\operatorname{tr} A}$

Proof: First consider the case of a $k \times k$ Jordan block:

$$\exp \operatorname{tr} \begin{pmatrix} c & 1 & 0 & \cdots & 0 & 0 \\ 0 & c & 1 & \cdots & 0 & 0 \\ 0 & 0 & c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & 1 \\ 0 & 0 & 0 & \cdots & 0 & c \end{pmatrix} = e^{ck}$$

$$\det \exp \begin{pmatrix} c & 1 & 0 & \cdots & 0 & 0 \\ 0 & c & 1 & \cdots & 0 & 0 \\ 0 & 0 & c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & 1 \\ 0 & 0 & 0 & \cdots & 0 & c \end{pmatrix} = \det \begin{pmatrix} e^c & e & 0 & \cdots & 0 & 0 \\ 0 & e^c & e & \cdots & 0 & 0 \\ 0 & 0 & e^c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^c & e \\ 0 & 0 & 0 & \cdots & 0 & e^c \end{pmatrix} = (e^c)^k = e^{ck}.$$

Therefore the theorem is true for Jordan blocks. Next, suppose we have a matrix in Jordan canonical form:

$$J = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \cdots & J_m \end{pmatrix}.$$

Then

$$\exp J = \begin{pmatrix} \exp J_1 & 0 & 0 & \cdots & 0 \\ 0 & \exp J_2 & 0 & \cdots & 0 \\ 0 & 0 & \exp J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \cdots & \exp J_m \end{pmatrix}$$

and so

$$\begin{aligned} \det \exp J &= (\det \exp J_1)(\det \exp J_2) \cdots (\det \exp J_n) \\ &= e^{\operatorname{tr} J_1} e^{\operatorname{tr} J_2} \cdots e^{\operatorname{tr} J_n} \\ &= e^{\operatorname{tr} J_1 + \operatorname{tr} J_2 + \cdots \operatorname{tr} J_n} \\ &= e^{\operatorname{tr} J}. \end{aligned}$$

Finally, given A in $\mathrm{M}(n,\mathbb{C})$, write $A = SJS^{-1}$, where J is in Jordan canonical form. Then

$$\begin{aligned}\det \exp A = \det \exp(SJS^{-1}) &= \det\big(S(\exp J)S^{-1}\big) \\ &= \det \exp J = e^{\operatorname{tr} J} = e^{\operatorname{tr}(SJS^{-1})} = e^{\operatorname{tr} A}. \quad \square\end{aligned}$$

Let V be a complex vector space equipped with an *inner product*. This is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all elements v, w, and u in V and all complex numbers α and β,

- $\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, u \rangle$;
- $\langle v, \alpha w + \beta u \rangle = \overline{\alpha} \langle v, w \rangle + \overline{\beta} \langle v, u \rangle$;
- $\langle w, v \rangle = \overline{\langle v, w \rangle}$;
- $\langle v, v \rangle \geq 0$, with $\langle v, v \rangle = 0$ if and only if $v = 0$.

An *orthonormal basis* for V is a vector space basis $\{e_k\}_{k=1}^n$ for V with the additional properties

- $\langle e_k, e_k \rangle = 1$ for $1 \leq k \leq n$;
- $\langle e_k, e_\ell \rangle = 0$ for $k \neq \ell$.

Let A be a linear transformation of V. Then

$$\operatorname{tr} A = \sum_{k=1}^{n} \langle Ae_k, e_k \rangle$$

and

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) \left\langle Ae_1, e_{\sigma(1)} \right\rangle \left\langle Ae_2, e_{\sigma(2)} \right\rangle \cdots \left\langle Ae_n, e_{\sigma(n)} \right\rangle.$$

These quantities are independent of the choice of orthonormal basis.

The *adjoint* of A is the linear transformation determined by the equation

$$\langle Av, w \rangle = \langle v, A^* w \rangle$$

for all v and w in V.

If we write A as a matrix with respect to an orthonormal basis, then A^* is the complex conjugate transpose of A; i.e., the (i,j) entry of A^* is $\overline{a_{ji}}$. Thus

$$\operatorname{tr} A^* = \overline{\operatorname{tr} A}, \qquad \det A^* = \overline{\det A}.$$

Now let V be an infinite-dimensional complex inner product space and define a norm $\|v\| := \sqrt{\langle v, v\rangle}$ for every v in V. We say that V is *complete* if every Cauchy sequence with respect to this norm is convergent. In this case we will use the letter $\mathcal{H}$ to denote our complex inner product space, and we call it a *Hilbert space*.

We will only consider *separable* Hilbert spaces. This means that $\mathcal{H}$ contains a countably infinite subset $\{e_k\}$ with the following properties:

- $\langle e_k, e_k\rangle = 1$ for all k;
- $\langle e_k, e_\ell\rangle = 0$ for $k \neq \ell$;
- $v = \sum_{k=1}^{\infty} \langle v, e_k\rangle e_k$ for every v in $\mathcal{H}$.

Warning: the set $\{e_k\}$ is **not** a vector space basis!

Let A be a linear transformation of $\mathcal{H}$. We say that A is *bounded* if

$$\|A\| := \sup\left\{\frac{\|Av\|}{\|v\|} : v \neq 0\right\} < \infty.$$

We will call a bounded linear transformation of $\mathcal{H}$ an *operator* on $\mathcal{H}$.

The collection of all operators on $\mathcal{H}$ is an *algebra* (closed under addition, multiplication [composition], scalar multiplication), and is denoted $\mathcal{B}(\mathcal{H})$.

How do we define trace for operators on $\mathcal{H}$?

Naive idea: choose an orthonormal basis $\{e_k\}$ for $\mathcal{H}$ and set

$$\operatorname{tr} A = \sum_{k=1}^{\infty} \langle Ae_k, e_k\rangle\,.$$

Problem 1: The right-hand side does not necessarily converge.

Example:

$$\operatorname{tr} I = \sum_{k=1}^{\infty} \langle Ie_k, e_k\rangle = \sum_{k=1}^{\infty} \langle e_k, e_k\rangle = \sum_{k=1}^{\infty} 1 = \infty.$$

So not every operator has a well-defined trace.

Problem 2: Even if the right-hand side does converge, its value may depend on the choice of orthonormal basis.

An operator P on $\mathcal{H}$ is *positive* if $\langle Pv, v\rangle \geq 0$ for all v in $\mathcal{H}$.

Example: Let A be any operator on $\mathcal{H}$. Then A^*A is positive, because

$$\langle A^*Av, v\rangle = \langle Av, Av\rangle \geq 0.$$

In fact, every positive operator P has this form for some operator A.

If P is positive, then $\sum_{k=1}^{\infty} \langle Pe_k, e_k\rangle$ is in $[0, \infty]$ and is independent of the choice of orthonormal basis.

Every positive operator P has a positive *square root operator* $\sqrt{P}$. Define

$$|A| := \sqrt{A^*A}.$$

Example: Take

$$A = \begin{pmatrix} -\frac{27}{25} + \frac{32}{25}i & -\frac{36}{25} - \frac{24}{25}i \\ -\frac{36}{25} - \frac{24}{25}i & -\frac{48}{25} + \frac{18}{25}i \end{pmatrix}.$$

Then

$$A^*A = \begin{pmatrix} \frac{29}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{36}{5} \end{pmatrix}.$$

Let

$$S = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

Then

$$S^{-1}(A^*A)S = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix},$$

whence

$$\sqrt{S^{-1}(A^*A)S} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

and thus

$$|A| = S\left(\sqrt{S^{-1}(A^*A)S}\right)S^{-1} = \begin{pmatrix} \frac{59}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{66}{25} \end{pmatrix}.$$

Define

$$\mathcal{L}^1(\mathcal{H}) := \left\{ A \in \mathcal{B}(\mathcal{H}) : \sum_{k=1}^{\infty} \langle |A| e_k, e_k \rangle < \infty \right\}.$$

The set $\mathcal{L}^1(\mathcal{H})$ is an ideal in $\mathcal{B}(\mathcal{H})$ and is called the *ideal of trace-class operators* on $\mathcal{H}$. For A in $\mathcal{L}^1(\mathcal{H})$ we can define $\operatorname{tr} A$ in the naive way we originally proposed:

$$\operatorname{tr} A = \sum_{k=1}^{\infty} \langle A e_k, e_k \rangle .$$

Properties of tr:

- $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$ for A and B in $\mathcal{L}^1(\mathcal{H})$;
- $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for A in $\mathcal{L}^1(\mathcal{H})$ and B in $\mathcal{B}(\mathcal{H})$;
- $\operatorname{tr}(SAS^{-1}) = \operatorname{tr} A$ for A in $\mathcal{L}^1(\mathcal{H})$ and S in $\mathcal{B}(\mathcal{H})$ invertible;
- $\operatorname{tr} A$ is the sum of the eigenvalues of A for all A in $\mathcal{L}^1(\mathcal{H})$.

Remark: This last statement, known as Lidskii's theorem, was not proved until 1959.

How do we define the determinant?

For $\|A\| < 1$, we can define the logarithm of $I + A$ by the infinite series

$$\log(I + A) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^n.$$

If A is trace class, then for $\mu \in \mathbb{C}$ with sufficiently small modulus, the operator $\log(1 + \mu A)$ is also trace class, so we can define

$$\det(I + \mu A) = e^{\operatorname{tr}(\log(I + \mu A))}$$

and then extend by analytic continuation, so that the domain of det is

$$\mathrm{GL}(1, (I + \mathcal{L}^1(\mathcal{H})),$$

the multiplicative group of invertible elements of $\mathcal{B}(\mathcal{H})$ of the form $I + L$ for some L in $\mathcal{L}^1(\mathcal{H})$.

Properties of det:

- $\det(AB) = (\det A)(\det B)$ for A and B in $\mathrm{GL}(1, I + \mathcal{L}^1(\mathcal{H}))$;
- $\det A^{-1} = (\det A)^{-1}$ for A in $\mathrm{GL}(1, (I + \mathcal{L}^1(\mathcal{H}))$;
- $\det(SAS^{-1}) = \det A$ for A in $\mathrm{GL}(1, (I+\mathcal{L}^1(\mathcal{H}))$ and S in $\mathcal{B}(\mathcal{H})$ invertible;
- $\det A$ is the product of the eigenvalues of A for A in $\mathrm{GL}(1, I + \mathcal{L}^1(\mathcal{H}))$.

These quantities are hard to compute directly, especially the determinant! However, in certain cases of geometric and/or topological interest, there are other ways to proceed.

Example 1:

Suppose $K : [a, b] \times [a, b] \to \mathbb{C}$ is continuous and define A in $\mathcal{B}(L^2[a, b])$ by the formula

$$(Af)(x) = \int_a^b K(x, y) f(y)\, dy.$$

This is an example of a *compact* operator. It is not always trace class (in fact, it is an open problem to find necessary and sufficient conditions on K so that A is trace class), but if A, is trace class, then

$$\operatorname{tr} A = \int_a^b K(x, x)\, dx.$$

We can also express $\det(I + A)$ in terms of K. For each n-tuple $(x_1, x_2, \ldots, x_n)$ in $[a, b]$, define

$$K_n(x_1, x_2, \ldots, x_n) = \det \begin{pmatrix} K(x_1, x_1) & K(x_1, x_2) & \cdots & K(x_1, x_n) \\ K(x_2, x_1) & K(x_2, x_2) & \cdots & K(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, x_1) & K(x_n, x_2) & \cdots & K(x_n, x_n) \end{pmatrix}$$

Then

$$\det(I + A) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_a^b \cdots \int_a^b K_n(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n.$$

Example 2:
Consider the Hilbert space $L^2(S^1)$ with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta.$$

This Hilbert space has orthonormal basis

$$\{e^{in\theta} : n \in \mathbb{Z}\} = \{z^n : n \in \mathbb{Z}\}.$$

Let $C(S^1)$ denote the algebra of continuous complex-valued functions on the circle. For each ϕ in $C(S^1)$, define an operator M_ϕ on $L^2(S^1)$ via pointwise multiplication:

$$(M_\phi f)(x) = \phi(x)f(x).$$

Next, let $H^2(S^1)$ be the Hilbert subspace of $L^2(S^1)$ whose orthonormal basis is

$$\{z^n : n \geq 0\}.$$

An alternate description of $H^2(S^1)$ is the Hilbert subspace of the elements of $L^2(S^1)$ that extend to analytic functions on the disk $\{z \in \mathbb{C} : |z| < 1\}$.

Define the *orthogonal projection* $P : L^2(S^1) \to H^2(S^1)$ by

$$P\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n.$$

Then for each ϕ in $C(T)$, define the *Toeplitz operator* T_ϕ on $H^2(S^1)$ by the formula

$$T_\phi = PM_\phi.$$

Properties of Toeplitz operators: For ϕ and ψ in $C(S^1)$ and λ in $\mathbb{C}$,

- $T_{\phi+\psi} = T_\phi + T_\psi$;
- $T_{\lambda\phi} = \lambda T_\phi$;
- $T_\phi^* = T_{\overline{\phi}}$.

$T_{\phi\psi} \neq T_\phi T_\psi$ in general, but for ϕ and ψ in $C^\infty(S^1)$, we have

$$T_\phi T_\phi - T_\psi T_\phi \in \mathcal{L}^1(\mathcal{H}).$$

Surprisingly (at first), the trace of this quantity can be nonzero. This is because $T_\phi T_\phi$ and $T_\psi T_\phi$ are typically not trace class operators, even though their difference is.

Example:

$$T_{z^{-3}}T_{z^3}(z^n) = z^n \text{ for all } n \geq 0$$

$$T_{z^3}T_{z^{-3}}(z^n) = \begin{cases} 0 & 0 \leq n < 3 \\ z^n & n \geq 3 \end{cases}$$

Therefore

$$\operatorname{tr}(T_{z^{-3}}T_{z^3} - T_{z^3}T_{z^{-3}}) = 3.$$

In general,

$$\operatorname{tr}(T_{z^m}T_{z^n} - T_{z^n}T_{z^m}) = \begin{cases} n & \text{if } m+n=0 \\ 0 & \text{otherwise.} \end{cases}$$

Also observe that

$$\frac{1}{2\pi i}\int_0^{2\pi} e^{im\theta} d(e^{in\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} ine^{im\theta}e^{in\theta}\, d\theta = \begin{cases} n & \text{if } m+n=0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem: For ϕ and ψ in $C^\infty(S^1)$,

$$\operatorname{tr}(T_\phi T_\psi - T_\psi T_\phi) = \frac{1}{2\pi i}\int_{S^1} \phi\, d\psi.$$

Proof: Write ϕ and ψ in terms of the basis $\{z^n : n \geq 0\}$ and combine the linearity of the trace and the integral with the computations in the example above. □

We can generalize this result somewhat. Define

$$\mathcal{T}^\infty := \left\{T_\phi + L : \phi \in C^\infty(S^1), L \in \mathcal{L}^1(H^2(S^1))\right\}.$$

Then there exists a short exact sequence

$$0 \longrightarrow \mathcal{L}^1(H^2(S^1)) \longrightarrow \mathcal{T}^\infty \xrightarrow{\sigma} C^\infty(S^1) \longrightarrow 0\ ,$$

and the *symbol map* $\sigma : \mathcal{T}^\infty \to C^\infty(S^1)$ is given by the formula $\sigma(T_\phi + L) = \phi$.

Theorem: For T_1 and T_2 in $\mathcal{T}^\infty$,

$$\operatorname{tr}(T_1T_2 - T_2T_1) = \frac{1}{2\pi i}\int_{S^1} \sigma(T_1)\, d(\sigma(T_2)).$$

Proof: Write $T_1 = T_{\phi_1} + L_1$ and $T_1 = T_{\phi_2} + L_2$. Then

$$\begin{aligned}
\operatorname{tr}(T_1T_2 - T_2T_1) &= \operatorname{tr}((T_{\phi_1} + L_1)(T_{\phi_2} + L_2) - (T_{\phi_2} + L_2)(T_{\phi_1} + L_1)) \\
&= \operatorname{tr}(T_{\phi_1}T_{\phi_2} - T_{\phi_2}T_{\phi_1} + T_{\phi_1}L_2 - L_2T_{\phi_1} \\
&\qquad\qquad + L_1T_{\phi_2} - T_{\phi_2}L_1 + L_1L_2 - L_2L_1) \\
&= \operatorname{tr}(T_{\phi_1}T_{\phi_2} - T_{\phi_2}T_{\phi_1}) + \operatorname{tr}(T_{\phi_1}L_2 - L_2T_{\phi_1}) \\
&\qquad\qquad + \operatorname{tr}(L_1T_{\phi_2} - T_{\phi_2}L_1) + \operatorname{tr}(L_1L_2 - L_2L_1) \\
&= \operatorname{tr}(T_{\phi_1}T_{\phi_2} - T_{\phi_2}T_{\phi_1}) \\
&= \frac{1}{2\pi i}\int \phi_1\, d\phi_2 \\
&= \frac{1}{2\pi i}\int \sigma(T_1)\, d(\sigma(T_2)).
\end{aligned}$$

Note that $\operatorname{tr}(T_1T_2 - T_2T_1)$ only depends on the symbols of T_1 and T_2!

Now let's look at the determinant.

Suppose ϕ, ψ are nowhere-vanishing functions in $C^\infty(S^1)$ and that the winding numbers of ϕ and ψ are zero. Then T_ϕ and T_ψ are invertible (this is a nontrivial fact!).

Warning: $T_\phi^{-1} \neq T_{\phi^{-1}}$ in general!

Note that

$$\sigma\left(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}\right) = \phi\psi\phi^{-1}\psi^{-1} = 1,$$

whence $T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}$ is in $I + \mathcal{L}^1(H^2(S^1))$.

$$\det\left(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}\right) = ??$$

It's not too hard to prove that the quantity we are taking the determinant of only depends on the symbols ϕ and ψ. That is, if T_1 and T_2 are invertible Toeplitz operators with $\sigma(T_1) = \phi$ and $\sigma(T_2) = \psi$, then

$$\det\left(T_1 T_2 T_1^{-1} T_2^{-1}\right) = \det\left(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}\right).$$

Theorem [Campbell-Baker-Hausdorff-Dynkin-$\cdots$]: Suppose A and B are operators on $\mathcal{H}$. If $\|A\| + \|B\| < \sqrt{2}$, then $(\exp A)(\exp B) = \exp C$, where

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] \\ + \text{ terms involving higher commutators of } A \text{ and } B.$$

Corollary:

$$(\exp A)(\exp B)(\exp(-A))(\exp(-B)) = \\ \exp\left([A, B] + \text{ terms involving higher commutators of } A \text{ and } B\right).$$

Suppose ϕ and ψ are close to 1. Then $\phi = e^\alpha$ and $\psi = e^\beta$ for functions α and β in $C^\infty(S^1)$. Note that

$$\sigma(\exp T_\alpha) = e^{\sigma(T_\alpha)} = e^\alpha = \phi$$
$$\sigma(\exp T_\beta) = e^{\sigma(T_\beta)} = e^\alpha = \psi.$$

Therefore

$$\begin{aligned}
\det\left(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}\right) &= \det\left(\exp(T_\alpha)\exp(T_\beta)\exp(-T_\alpha)\exp(-T_\beta)\right) \\
&= \det\exp\left(T_\alpha T_\beta - T_\beta T_\alpha + \text{ higher commutators}\right) \\
&= \exp\operatorname{tr}\left(T_\alpha T_\beta - T_\beta T_\alpha + \text{ higher commutators}\right) \\
&= \exp\left(\frac{1}{2\pi i}\int_{S^1} \alpha\, d\beta\right) \\
&= \exp\left(\frac{1}{2\pi i}\int_{S^1} \log\phi\, d(\log\psi)\right) \\
&= \exp\left(\frac{1}{2\pi i}\int_{S^1} \log\phi \cdot \frac{d\psi}{\psi}\right).
\end{aligned}$$

Let's look at this from a different point of view.

Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{H}^n$ is also a Hilbert space:

$$\langle (v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_n) \rangle := \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle + \cdots + \langle v_n, w_n \rangle \, .$$

We can view elements of $\mathcal{B}(\mathcal{H}^n)$ as elements of $\mathrm{M}(n, \mathcal{B}(\mathcal{H}))$. By extending the notion of symbol in the obvious way, we have a short exact sequence

$$0 \longrightarrow \mathcal{L}^1((H^2(S^1))^n) \longrightarrow \mathrm{M}(n, \mathcal{T}^\infty) \xrightarrow{\sigma} \mathrm{M}(n, C^\infty(S^1)) \longrightarrow 0 \, .$$

Suppose ϕ and ψ are arbitrary invertible elements of $C^\infty(S^1)$. Then we can find matrices R and S in $\mathrm{GL}(3, \mathcal{T}^\infty)$ such that

$$\sigma(R) = \begin{pmatrix} \phi & 0 & 0 \\ 0 & \phi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\sigma(S) = \begin{pmatrix} \psi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \psi^{-1} \end{pmatrix} .$$

For example, we can choose

$$R = \begin{pmatrix} 2T_\phi - T_\phi T_{\phi^{-1}} T_\phi & T_\phi T_{\phi^{-1}} - I & 0 \\ I - T_{\phi^{-1}} T_\phi & T_{\phi^{-1}} & 0 \\ 0 & 0 & I \end{pmatrix}$$

and

$$S = \begin{pmatrix} 2T_\psi - T_\psi T_{\psi^{-1}} T_\psi & 0 & T_\psi T_{\psi^{-1}} - I \\ 0 & I & 0 \\ I - T_{\psi^{-1}} T_\psi & 0 & T_{\psi^{-1}} \end{pmatrix}$$

We infer from the short exact sequence above that the operator $RSR^{-1}S^{-1}$ is determinant-class. Furthermore, the value of this determinant does not depend on the choice of R and S satisfying the properties above - the determinant of $RSR^{-1}S^{-1}$ only depends on ϕ and ψ.

Suppose that ϕ and ψ are restrictions of meromorphic functions (which we also denote ϕ and ψ) defined in a neighborhood of the closed unit disk such that neither ϕ nor ψ has zeros or poles on the unit circle. For each point z in the open unit disk $\mathbb{D}$, define

$$v(\phi, z) = \begin{cases} m & \text{if } \phi \text{ has a zero of order } m \text{ at } z \\ -m & \text{if } \phi \text{ has a pole of order } m \text{ at } z \\ 0 & \text{if } \phi \text{ has neither a zero nor a pole at } z, \end{cases}$$

and similarly define $v(\psi, z)$. The quantity

$$\lim_{w\to z} (-1)^{v(\phi,z)v(\psi,z)} \frac{\psi(w)^{v(\phi,z)}}{\phi(w)^{v(\psi,z)}}$$

is called the *tame symbol* of ϕ and ψ at z and is denoted $(\phi, \psi)_z$.

Example:

$$\phi(z) = \frac{z^3 - 3z^2}{2z+1} \quad \text{double zero at 0, simple zero at 3, simple pole at } -1/2$$

$$\psi(z) = \frac{2z-1}{z^3} \quad \text{simple zero at } 1/2\text{, triple pole at 0}$$

$$\begin{aligned}(\phi,\psi)_0 &= \lim_{w\to 0}\left((-1)^{(2)(-3)} \frac{\left(\frac{2w-1}{w^3}\right)^2}{\left(\frac{w^2(w-3)}{2w+1}\right)^{-3}}\right)\\ &= \lim_{w\to 0} \frac{(2w-1)^2}{w^6}\cdot\frac{w^6(w-3)^3}{(2w+1)^3}\\ &= \lim_{w\to 0}\frac{(2w-1)^2(w-3)^3}{(2w+1)^3}\\ &= -27\end{aligned}$$

$$\begin{aligned}(\phi,\psi)_{-1/2} &= \lim_{w\to -1/2}\left((-1)^{(-1)(0)} \frac{\left(\frac{2w-1}{w^3}\right)^{-1}}{\left(\frac{w^2(w-3)}{2w+1}\right)^{0}}\right)\\ &= \lim_{w\to -1/2}\frac{w^3}{2w-1}\\ &= \frac{1}{16}\end{aligned}$$

$$\begin{aligned}(\phi,\psi)_{1/2} &= \lim_{w\to 1/2}\left((-1)^{(0)(-1)} \frac{\left(\frac{2w-1}{w^3}\right)^{0}}{\left(\frac{w^2(w-3)}{2w+1}\right)^{1}}\right)\\ &= \lim_{w\to 1/2}\frac{2w+1}{w^2(w-3)}\\ &= -\frac{16}{5}\end{aligned}$$

We will not compute $(\phi, \psi)_3$ for reasons that will be become clear in a minute. For all other complex numbers z, we see that $(\phi, \psi)_z = 1$.

Theorem:

$$\det(RSR^{-1}S^{-1}) = \prod_{z \in \mathbb{D}} (\phi, \psi)_z^{-1}.$$

Remark 1: Suppose that T_Φ and T_ψ are invertible. Then we can take

$$R = \begin{pmatrix} T_\phi & 0 & 0 \\ 0 & T_\phi^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}$$

and

$$S = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T_\psi^{-1} \end{pmatrix},$$

whence

$$\det(RSR^{-1}S^{-1}) = \det \begin{pmatrix} T_\phi T_\psi T_\phi^{-1} T_\psi^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \det \left(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1} \right).$$

Remark 2: In fact, $\det(RSR^{-1}S^{-1})$ only depends on the *Steinberg symbol* $\{\phi, \psi\}$ of ϕ and ψ. This is an element of the algebraic K-theory group $K_2(C^\infty(S^1))$, and we can use the above theorem to prove that certain Steinberg symbols are nontrivial.

Surprising fact that comes out of this circle of ideas: if both ϕ and $\psi := 1 - \phi$ are invertible, then $\det(RSR^{-1}S^{-1}) = 1$.

The de la Harpe-Skandalis "Determininant"

Suppose A is a unital Banach algebra with a trace $\tau : A \to \mathbb{C}$. Then we can extend τ to a trace on $\mathrm{M}(n, A)$ in the obvious way.

Let $\mathrm{GL}_0(n, A)$ denote the connected component of the identity matrix in $\mathrm{GL}(n, A)$. Then given a C^1-path ξ in $\mathrm{GL}_0(n, A)$, define

$$\widetilde{\Delta}(\xi) = \tau\left(\frac{1}{2\pi i}\int_0^1 \xi'(t)\xi(t)^{-1}\,dt\right) = \frac{1}{2\pi i}\int_0^1 \tau\left(\xi'(t)\xi(t)^{-1}\right)\,dt.$$

Properties of $\widetilde{\Delta}$:

- Suppose that $\xi = \xi_1 \cdot \xi_2$ (pointwise product). Then $\widetilde{\Delta}(\xi) = \widetilde{\Delta}(\xi_1) + \widetilde{\Delta}(\xi_2)$;

- If $|\xi(t) - 1\| < 1$ for all $0 \leq t \leq 1$, then

$$\widetilde{\Delta}(\xi) = \frac{1}{2\pi i}\tau\left(\log(\xi(1)) - \frac{1}{2\pi i}\log\xi((0))\right);$$

- The value of $\widetilde{\Delta}(\xi)$ only depends on the homotopy class of ξ *with the end-points fixed*;

- Given an idempotent p (i.e, $p^2 = p$) in $\mathrm{M}(n, A)$, define ξ_p by the formula $\xi_p(t) = e^{2\pi i t}p + (1 - p)$. Then $\widetilde{\Delta}(\xi_p) = \tau(p)$.

Suppose x is an element of $\mathrm{GL}_0(n, A)$, choose a C^1-path ξ from 1 to x, and define $\Delta(x) = \widetilde{\Delta}(\xi)$.

Problem: $\Delta(x)$ depends on the choice of path ξ.

What we really have, via the (in)famous Bott periodicity theorem in K-theory, is a function into $\mathbb{C}/\tau(K_0(A))$.

Properties of Δ:

- Δ is a group homomorphism;

- Δ is surjective if and only if τ is surjective;

- $\Delta(e^y) = \tau(y) + \tau(K_0(A))$ for y in $\mathrm{M}(n, A)$.

Corollary: Suppose that $\tau(K_0(A) \cong \mathbb{Z})$. Then

$$\exp(2\pi i\Delta) : \mathrm{GL}_0(n, A) \to \mathbb{C}^*$$

is a group homomorphism, and

$$\exp(2\pi i\Delta)(e^y) = e^{\tau(y)}$$

for y in $\mathrm{M}(n, A)$. In particular, if $A = \mathbb{C}$ and τ is the identity map, then $\exp(2\pi i\Delta)$ is the usual determinant on $\mathrm{M}(n, \mathbb{C})$.

The Fuglede-Kadison-Brown-Hochs-Kaad-Schemaitat Determinant

A *von Neumann algebra* is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the topology of pointwise convergence.

Example: $L^\infty(\mathbb{R})$

Suppose we have a "nice" (normal, faithful, semifinite) trace τ defined on positive elements in $\mathcal{N}$.

Example: For $L^\infty(\mathbb{R})$, take $\tau(f) = \displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.

Let $\mathcal{L}^1(\mathcal{N})$ be the trace ideal. For invertible elements in $\mathcal{N}$ of the form $1+x$ with x in $\mathcal{L}^1(\mathcal{N})$, the aforementioned authors define a determinant homomorphism $\det_\tau$ with values in $(0, \infty)$:

$$\det_\tau(1 + x) = e^{\tau(\log|1+x|)}.$$

This determinant is multiplicative – this is highly nontrivial to show and involves techniques from algebraic K-theory and Connes' cyclic homology.

Example: Wiener-Hopf Operators

Consider the Hilbert space $L^2(\mathbb{R})$. There is a Hilbert subspace $H^2(\mathbb{R})$ of $L^2(\mathbb{R})$ that consists of elements that have an analytic extension to the upper half plane, satisfying a certain growth condition. Let P denote orthogonal projection from $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$.

Let $C_b(\mathbb{R})$ denote the algebra of bounded continuous functions on the real line. For each ϕ in $C_b(\mathbb{R})$, multiplication by ϕ defines an element of $\mathcal{B}(L^2(\mathbb{R}))$, and we can compress to $H^2(\mathbb{R})$ just as we did in the circle case to obtain an operator W_ϕ on $\mathcal{B}(H^2(\mathbb{R}))$:

$$W_\phi = PM_\phi.$$

The algebra of *almost periodic* functions on $\mathbb{R}$ is the norm-closed subalgebra of $C_b(\mathbb{R})$ generated by the functions $t \to e^{i\lambda t}$ for λ real. This algebra can be identified with $C(\mathbb{R}_B)$, the continuous functions on the *Bohr compactification* of $\mathbb{R}$.

Let $\mathcal{W}$ be the C^*-subalgebra of $\mathcal{B}(H^2(\mathbb{R}))$ generated $\{W_\phi : \phi \in C(\mathbb{R}_B)\}$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{W} \xrightarrow{\sigma} C(\mathbb{R}_B) \longrightarrow 0,$$

where $\mathcal{C}$ is the commutator ideal of $\mathcal{W}$. This commutator ideal lives inside, but is not equal to, the trace ideal associated to a von Neumann trace. Just as in the circle case, there is a "smooth" version of this short exact sequence:

$$0 \longrightarrow \mathcal{C}^\infty \longrightarrow \mathcal{W}^\infty \xrightarrow{\sigma} C^\infty(\mathbb{R}_B) \longrightarrow 0.$$

Theorem: Suppose W_1 and W_2 in $\mathcal{W}^\infty$ have symbols ϕ and ψ respectively. Then

$$\operatorname{tr}(W_1W_2 - W_2W_1) = \lim_{R\to\infty} \left(\frac{1}{2R} \int_{-R}^{R} \phi(t)\psi'(t)\, dt \right).$$

Theorem: Suppose W_1 and W_2 in $\mathcal{W}^\infty$ are invertible, have symbols ϕ and ψ respectively, and are close to I. Then

$$\det_\tau \left(W_1W_2W_1^{-1}W_2^{-1}\right) = \lim_{R\to\infty} \left(\exp\left(\frac{1}{2R} \operatorname{Re}\left(\int_{-R}^{R} \log(\phi(t)) \frac{\psi'(t)}{\psi(t)}\, dt \right)\right)\right).$$