Polynomial vector fields in $\mathbb{R}^3$ having a quadric of revolution as an invariant algebraic surface

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Plan of the talk:

1. **Introduction and generalities;**
2. Our results on quadrics of revolution;
3. Outline of the proofs;
4. Open problem.

1. Introduction

A **polynomial differential system in $\mathbb{R}^2$** is a system of the form

$$x' = \frac{dx}{dt} = P_1(x, y), \quad y' = \frac{dy}{dt} = P_2(x, y), \tag{1}$$

where $P_i \in \mathbb{R}[x, y]$ for $i = 1, 2$ and t is the independent variable. We denote by

$$\mathcal{X}(x, y) = (P_1(x, y), P_2(x, y)), \tag{2}$$

the polynomial vector field associated to system (1).

We say that $m = \max\{m_i\}$, where m_i is the degree of P_i, $i = 1, 2$, is the **degree** of the polynomial differential system (1), or of the polynomial vector field (2).

1. Introduction: 16th Hilbert Problem

One of the most difficult objects to control in the qualitative theory of ordinary differential equations in dimension two are the **limit cycles**: maximum number and distribution.

This problem is related with the second part of the 16th Hilbert Problem.

D. HILBERT, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. Paris, Nachr. Ges. Wiss. Göttingen Math. Phys. KL., 253–297 (1900). English transl., Bull. Amer. Math. Soc., **8** (1902), 437–479.

1. Introduction: 16th Hilbert Problem

Problem 1 (Second part of 16th Hilbert Problem)

Consider polynomial vector fields of **degree** m *in the plane.*

Prove (or disprove) that there exists a uniform upper bound, depending on the **degree** m *of the vector fields, for the number of limit cycles.*

1. Introduction: 16th Hilbert Problem

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential systems. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential system of the first order and degree of the form

$$\frac{dy}{dx} = \frac{Y}{X},$$

where X and Y are rational integral functions of the nth degree in x and y.

Second part of 16th Hilbert Problem.

1. Introduction: 16th Hilbert Problem

The following theorem is essential in the study of 16th Hilbert Problem.

Theorem 1 (Écalle, Ilyashenko)

A polynomial vector field in the plane has a finite number of limit cycles.

J. Écalle, *Introduction aux functions analysables et preuve constructive de la conjecture de Dulac*, Hermann, Paris, 1992.

Y. Ilyashenko, *Finiteness theorems for limit cycles*, American Mathematical Society, Providence, RI, 1993.

1. Introduction: 16th Hilbert Problem

Smale wrote the following comment.

Remark 1

"These two papers have yet to be thoroughly digested by the mathematical community".

S. SMALE, *Mathematical problems for the next century*, Math. Intelligencer, **20** (1998), 7–15.

By Theorem 1 we have the finiteness, but no bounds.

The simplest case $m = 2$: quadratic vector fields

In 1957, Petrovskii and Landis claimed that quadratic vector fields have at most 3 limit cycles.

I.G. PETROVSKII, E.M. LANDIS, *On the number of limit cycles of the equation $dy/dx = P(x, y)/Q(x, y)$, where P and Q are polynomials*, Mat. Sb. N.S., sc 43 (1957), 149–168 (Russian), and Amer. Math. Soc. Transl., 14 (1960), 181–200.

In 1959 a gap was found in the arguments of Petrovskii and Landis.

1. Introduction: Quadratic vector fields

Later, Chen and Wang in 1979 provided the first quadratic vector field having 4 limit cycles.

L.S. CHEN, M.S. WANG, *The relative position, and the number, of limit cycles of a quadratic differential system*, Acta Math. Sinica, **22** (1979), 751–758.

Up to now 4 is the maximum number of limit cycles known for a quadratic vector field.

1. Introduction: Quadratic vector fields

Bamon proved in 1986 that any quadratic vector field has finitely many limit cycles.

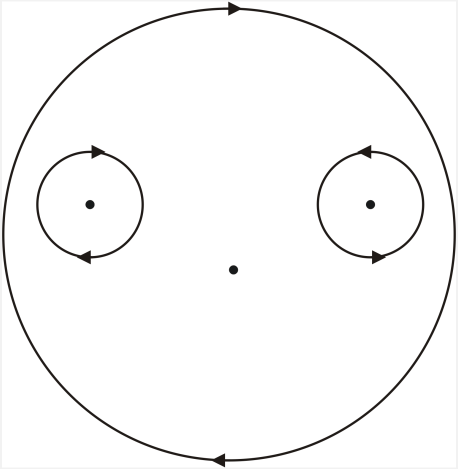
R. BAMON, *Quadratic vector fields in the plane have a finite number of limit cycles*, Int. Hautes Études Sci. Publ. Math., **64** (1986), 111–142.

1. Introduction: Quadratic vector fields

It can be proved the following properties of quadratic vector fields:

- A closed orbit is convex;
- There is a unique equilibrium point in the interior of a closed orbit;
- Two closed orbits have the same (resp. opposite) orientation if their interiors have (resp. do not have) common points.

1. Introduction: Quadratic vector fields



This distribution of limit cycles is not possible.

1. Introduction: Quadratic vector fields

So, the distribution of limit cycles of quadratic vector fields has only one or two nests.

A quadratic vector field has an (n_1, n_2)**–distribution** of limit cycles if it has n_1 limit cycles in one nest and n_2 limit cycles in the other, where n_1 and n_2 are non–negative integers.

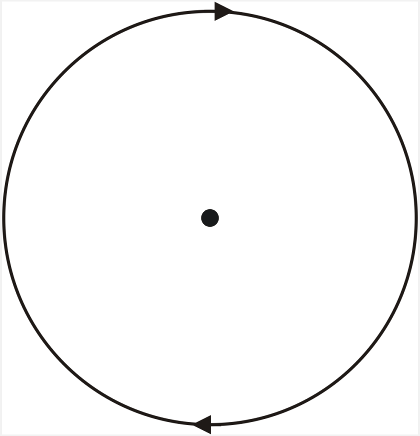
Zhang proved that in two nests case at least one nest contains a unique limit cycle.

P. ZHANG, *On the distribution and number of limit cycles for quadratic systems with two foci*, Qual. Theory Dyn. Syst., **3** (2002), 437–463.

Therefore, the $(2, 2)$–distribution of limit cycles for a quadratic vector field is impossible.

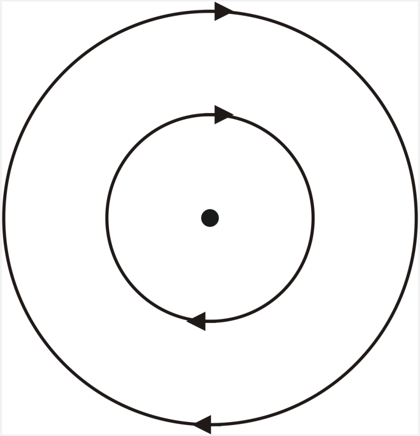
1. Introduction: Quadratic vector fields

Some known distributions of limit cycles:



$(1,0)$–distribution: $x' = y + y^2, \quad y' = -x + y/2 - xy + 3y^2/2.$

1. Introduction: Quadratic vector fields

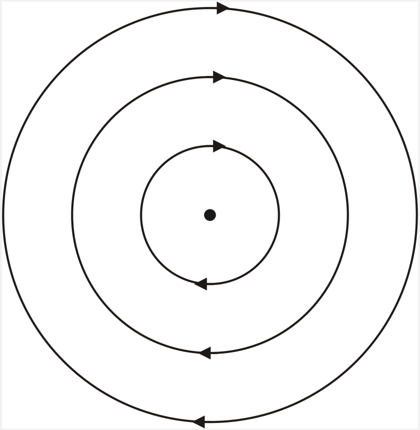


(2, 0)–distribution: $x' = 1 + xy, \quad y' = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + ay^2,$

with $a = 3/2$, $a_{11} = 4/5$, $a_{20} = -15$, $a_{10} = 9175/1000$, $a_{01} = 2a + 1 - a_{11}$ and

$$a_{00} + a_{10} - a_{01} + a_{20} - a_{11} + a_{02} = 0.$$

1. Introduction: Quadratic vector fields

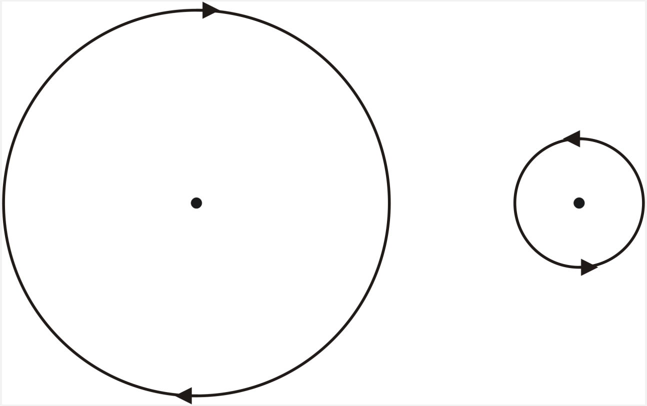


$(3,0)$–distribution: $x' = 1 + xy, \quad y' = a_{20}(x^2 - 1) + a_{10}(x - 1) + a_{11}(xy + 1) + a_{01}(y + 1) + a(y^2 - 1),$

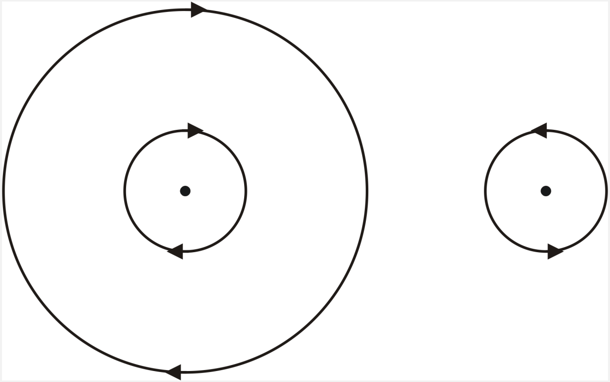
with $a = 7/5$, $a_{11} = 8012/10000$, $a_{20} = -15$, $a_{01} = 2998/1000$, $x_0 = -88/100$ and

$$a_{10} = -a_{20}(x_0 + 1) - a_{01}/x_0 + a(x_0 + 1)/x_0^2.$$

1. Introduction: Quadratic vector fields



$(1,1)$–distribution: $x' = y + y^2, \quad y' = -x/2 + y/2 - xy + 13y^2/10.$

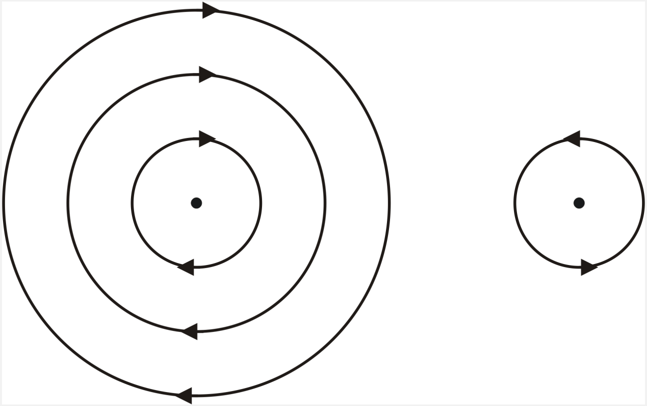


(2, 1)–distribution: $x' = 1 + xy, \quad y' = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + ay^2,$

with $a = 17/23$, $a_{11} = 221/115$, $a_{20} = -18$, $a_{10} = -54$, $a_{01} = 2a + 1 - a_{11}$ and

$$a_{00} + a_{10} - a_{01} + a_{20} - a_{11} + a_{02} = 0.$$

1. Introduction: Quadratic vector fields



(3, 1)–distribution: $x' = 1 + xy, \quad y' = a_{20}(x^2 - 1) + a_{10}(x - 1) + a_{11}(xy + 1) + a_{01}(y + 1) + a(y^2 - 1),$

with $a = 18/23$, $a_{11} = 17313/10000$, $a_{20} = -40$, $a_{01} = 835/1000$, $x_0 = -4$ and

$$a_{10} = -a_{20}(x_0 + 1) - a_{01}/x_0 + a(x_0 + 1)/x_0^2.$$

1. Introduction: 16th Hilbert Problem

Finally, there is an article under evaluation providing an answer to the 16th Hilbert Problem.

J. Llibre, P. Pedregal, *Hilbert's 16th Problem. When differential systems meet variational principles*, preprint, 2015.

The main theorem of this article is the following.

1. Introduction: 16th Hilbert Problem

Theorem 1. *An upper bound for the maximum number $H(n)$ of limit cycles that a polynomial differential system of degree $n > 1$ can have is*

$H(n) \leq 5n^3 - 13n^2 + 12n$ *if n is even, and* $H(n) \leq 5n^3 - 13n^2 + 10n$ *if n is odd.*

Plan of the talk:

1. Introduction and generalities;
2. **Our results on quadrics of revolution;**
3. Outline of the proofs;
4. Open problem.

2. Our results on quadrics of revolution

Now we study some results of the article

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When Parallels and Meridians are Limit Cycles for Polynomial Vector Fields on Quadrics of Revolution in the Euclidean 3-Space

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2. Our results on quadrics of revolution

We study polynomial vector fields of arbitrary degree in $\mathbb{R}^3$ with an invariant quadric of revolution.

We characterize all the possible configurations of invariant parallels that these vector fields can exhibit.

Furthermore we analyze when these invariant parallels can be limit cycles.

2. Our results on quadrics of revolution

As usual we denote by $\mathbb{K}[x, y, z]$ the ring of the polynomials in the variables x, y and z with coefficients in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. By definition a **polynomial differential system in $\mathbb{R}^3$** is a system of the form

$$\frac{dx}{dt} = P_1(x, y, z), \quad \frac{dy}{dt} = P_2(x, y, z), \quad \frac{dz}{dt} = P_3(x, y, z), \quad (3)$$

where $P_i \in \mathbb{R}[x, y, z]$ for $i = 1, 2, 3$ and t is the independent variable.

We denote by

$$\mathcal{X}(x, y, z) = (P_1(x, y, z), P_2(x, y, z), P_3(x, y, z)), \quad (4)$$

the polynomial vector field associated to system (3).

2. Our results on quadrics of revolution

We say that $m = \max\{m_i\}$, where m_i is the degree of P_i, $i = 1, 2, 3$, is the **degree** of the polynomial differential system (3), or of the polynomial vector field (4).

An **invariant algebraic surface** for system (3) or for the vector field (4) is an algebraic surface $M = f^{-1}(0)$ with $f \in \mathbb{R}[x, y, z]$, $f \neq 0$, such that for some polynomial $K \in \mathbb{R}[x, y, z]$ we have $\mathcal{X} f = Kf$.

The polynomial K is called the **cofactor** of the invariant algebraic surface $M = f^{-1}(0)$. We remark that if the polynomial system has degree m, then any cofactor has degree at most $m - 1$.

The name **invariant** comes from the fact that if a solution curve of system (3) has a point on the algebraic surface $M = f^{-1}(0)$, then the whole solution curve is contained in $M = f^{-1}(0)$.

2. Our results on quadrics of revolution

We consider polynomial vector fields $\mathcal{X}$ of degree $m > 1$ in $\mathbb{R}^3$ having a non–degenerate quadric of revolution

$$\mathbb{M}^2 = \mathcal{G}^{-1}(0)$$

as an invariant algebraic surface, that is

$$\mathcal{X}\mathcal{G} = K\mathcal{G},$$

where K is a polynomial of degree at most $m - 1$ and $\mathcal{G}$ defines one of the non–degenerate quadric of revolution that after an affine change of coordinates we can assume of the form:

2. Our results on quadrics of revolution

- **Cone:** $\mathcal{G}(x, y, z) = x^2 + y^2 - z^2$,
- **Cylinder:** $\mathcal{G}(x, y, z) = x^2 + y^2 - 1$,
- **One–sheet hyperboloid:** $\mathcal{G}(x, y, z) = x^2 + y^2 - z^2 - 1$,
- **Two–sheet hyperboloid:** $\mathcal{G}(x, y, z) = x^2 + y^2 - z^2 + 1$,
- **Paraboloid:** $\mathcal{G}(x, y, z) = x^2 + y^2 - z$,
- **Sphere:** $\mathcal{G}(x, y, z) = x^2 + y^2 + z^2 - 1$.

2. Our results on quadrics of revolution

On $\mathbb{M}^2$ we define **parallels** as the curves obtained by the intersection of $\mathbb{M}^2$ with the planes orthogonal to the z–axis.

More precisely, the parallels are obtained intersecting the planes $z = k$ (for suitable $k \in \mathbb{R}$) with $\mathbb{M}^2$.

2. Our results on quadrics of revolution: Paraboloid

Theorem 1

Let $\mathcal{X}$ be a polynomial vector field of degree $m > 1$ on the paraboloid.

Assume that $\mathcal{X}$ has finitely many invariant parallels.

The following statements hold.

2. Our results on quadrics of revolution: Paraboloid

(a) The number of invariant parallels of $\mathcal{X}$ is at most $m - 1$.

2. Our results on quadrics of revolution: Paraboloid

(b) Fix $1 \leq k \leq m-1$ and consider the vector field

$$\mathcal{X}(x,y,z) = (2y(x-2k) + h(z), -2x(x-2k), 2x\,h(z))\,, \quad (5)$$

on the paraboloid where

$$h(z) = \varepsilon z^{m-k-1} \prod_{i=1}^{k} (z-i), \quad \varepsilon > 0 \textit{ small}.$$

Then $\mathcal{X}$ has exactly k invariant parallels which are limit cycles.

These limit cycles are stable or unstable alternately.

Plan of the talk:

1. Introduction and generalities;
2. Our results on quadrics of revolution;
3. **Outline of the proofs;**
4. Open problem.

Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ and let W be an $\mathbb{R}$–vector subspace of $\mathbb{R}[x, y, z]$ of finite dimension $s > 1$.

Let $B = \{f_1, \ldots, f_s\}$ be a basis of W.

The **extactic polynomial of $\mathcal{X}$ associated to W** is the polynomial

$$\mathcal{E}_{W,B}(\mathcal{X}) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_s \\ \mathcal{X} f_1 & \mathcal{X} f_2 & \cdots & \mathcal{X} f_s \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{s-1} f_1 & \mathcal{X}^{s-1} f_2 & \cdots & \mathcal{X}^{s-1} f_s \end{pmatrix},$$

where

$$\mathcal{X}^j f_i = \mathcal{X}^{j-1}(\mathcal{X} f_i).$$

3. Outline of the proofs: Extactic polynomial

It follows from the properties of the determinant and of the derivation that the definition of extactic polynomial is independent of the chosen basis B of W in the following sense:

If we take another basis B' of W then

$$\mathcal{E}_{W,B'}(\mathcal{X}) = \alpha_{B,B'}\mathcal{E}_{W,B}(\mathcal{X}),$$

where

$$\alpha_{B,B'} \neq 0$$

is the determinant of the matrix of the change of basis.

So, from now on, we use the symbol $\mathcal{E}_W(\mathcal{X})$ not mentioning the specific basis of W.

3. Outline of the proofs: Extactic polynomial

Proposition 1

Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ and let W be a finite $\mathbb{R}$–vector subspace of $\mathbb{R}[x, y, z]$ with $\dim(W) = s > 1$.

Consider $f \in W$, $f \neq 0$, and suppose that $M = f^{-1}(0)$ is an algebraic invariant surface for the vector field $\mathcal{X}$. Then f is a factor of $\mathcal{E}_W(\mathcal{X})$.

3. Outline of the proofs: Proof of Proposition 1

By assumption $f \in W$. So consider f as the first element of a basis B of W, that is $B = \{f, f_2, \ldots, f_s\}$.

By assumption $M = f^{-1}(0)$ is an algebraic invariant surface for the vector field $\mathcal{X}$, that is there is a polynomial K such that

$$\mathcal{X} f = K f.$$

Claim:

$$\mathcal{X}^j(f) = K_j f,$$

where

$$K_1 = K, \quad K_j = \mathcal{X}(K_{j-1}) + K_{j-1}K, \quad j \geq 2.$$

In fact,

$$\mathcal{X}^1(f) = \mathcal{X}(f) = K\, f = K_1\, f.$$

$$\begin{aligned}\mathcal{X}^2(f) &= \mathcal{X}(\mathcal{X} f) = \mathcal{X}(K_1\, f) = \mathcal{X}(K_1)\, f + K_1 \mathcal{X}(f)\\ &= \mathcal{X}(K_1)\, f + K_1\, K\, f = (\mathcal{X}(K_1) + K_1\, K)\, f = K_2\, f.\end{aligned}$$

By induction on j, if $\mathcal{X}^j(f) = K_j\, f$ then

$$\begin{aligned}\mathcal{X}^{j+1}(f) &= \mathcal{X}(\mathcal{X}^j f) = \mathcal{X}(K_j\, f) = \mathcal{X}(K_j)\, f + K_j \mathcal{X}(f)\\ &= \mathcal{X}(K_j)\, f + K_j\, K\, f = (\mathcal{X}(K_j) + K_j\, K)\, f = K_{j+1}\, f.\end{aligned}$$

3. Outline of the proofs: Proof of Proposition 1

From the definition of an extactic polynomial we have

$$\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix} f & f_2 & \cdots & f_s \\ K\, f & \mathcal{X} f_2 & \cdots & \mathcal{X} f_s \\ \vdots & \vdots & \cdots & \vdots \\ K_{s-1}\, f & \mathcal{X}^{s-1} f_2 & \cdots & \mathcal{X}^{s-1} f_s \end{pmatrix}.$$

So each entry of the first column of the above matrix is a multiple of f. This implies that f is a factor of $\mathcal{E}_W(\mathcal{X})$ proving the proposition.

$\blacksquare$

In order to study invariant parallels we must consider the intersection of the planes $z - z_0 = 0$ (for suitable z_0) with $\mathbb{M}^2$ such that they are invariant by the flow of $\mathcal{X}$.

Consider W spanned by $B = \{f_1(x, y, z) = 1, f_2(x, y, z) = z\}$. Thus $g(x, y, z) = z - z_0 \in W$. By Proposition 1 it is necessary that g be a factor of the extactic polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X})$, which can be written as

$$\mathcal{E}_{\{1,z\}}(\mathcal{X}) = \det \begin{pmatrix} 1 & z \\ \mathcal{X}(1) & \mathcal{X}(z) \end{pmatrix} = \det \begin{pmatrix} 1 & z \\ 0 & P_3 \end{pmatrix} = P_3.$$

3. Outline of the proofs: Extactic polynomial

Proposition 2

Let $\mathcal{X}$ be the polynomial vector field (4)*. Consider W spanned by $B = \{f_1(x, y, z) = 1, f_2(x, y, z) = z\}$.*

If $g(x, y, z) = z - z_0$ is a factor of the extactic polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X})$, then $P_{z_0} = g^{-1}(0)$ is an invariant plane of $\mathcal{X}$.

3. Outline of the proofs: Proof of Proposition 2

By assumption $g(x, y, z) = z - z_0 \in W$ is a factor of the extactic polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X})$, that is g is a factor of

$$\mathcal{E}_{\{1,z\}}(\mathcal{X}) = \det \begin{pmatrix} 1 & z \\ \mathcal{X}(1) & \mathcal{X}(z) \end{pmatrix} = \det \begin{pmatrix} 1 & z \\ 0 & P_3 \end{pmatrix} = P_3.$$

So there is a polynomial K such that

$$\mathcal{E}_{\{1,z\}}(\mathcal{X}) = P_3 = K\, g.$$

From the definition of an invariant algebraic surface we have

$$\mathcal{X} g = (P_1, P_2, P_3) \cdot \nabla g = (P_1, P_2, P_3) \cdot (0, 0, 1) = P_3 = K\, g,$$

that is, the above polynomial K is a cofactor of the invariant algebraic surface $P_{z_0} = g^{-1}(0)$.

■

3. Outline of the proofs: Extactic polynomial

Remark 2

We remark that Propositions 1 and 2 transform the study of invariant parallels of a polynomial vector field $\mathcal{X} = (P_1, P_2, P_3)$ *of degree* $m > 1$ *on* $\mathbb{M}^2$ *into the study of factors of the form*

$$g(x, y, z) = z - z_0$$

of the extactic polynomial

$$\mathcal{E}_{\{1,z\}}(\mathcal{X}) = P_3.$$

3. Outline of the proofs: Averaging Theorem

The Averaging Theorem (due to Lagrange in his studies in Celestial Mechanics) gives a relation between the solutions of a non autonomous differential system and the solutions of an autonomous one, the averaged differential system. More details can be found in the following book

F. VERHULST, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer–Verlag, Berlin, 1990.

3. Outline of the proofs: Averaging Theorem

Theorem 2 (Averaging Theorem of First Order)

Consider the system

$$\dot{u}(t) = \frac{du(t)}{dt} = \varepsilon F(t, u(t)) + \varepsilon^2 R(t, u(t), \varepsilon). \tag{6}$$

Assume that the functions

$$F, \;\; R, \;\; D_u F, \;\; D_u^2 F, \;\; D_u R$$

are continuous and bounded by a constant M (independent of ε) in $[0, \infty) \times D \subset \mathbb{R} \times \mathbb{R}^n$ with $-\varepsilon_0 < \varepsilon < \varepsilon_0$.

Moreover, suppose that F and R are T–periodic in t, with T independent of ε.

(a) If $a \in D$ is a zero of the averaged function

$$f(u) = \frac{1}{T} \int_0^T F(s, u) ds, \tag{7}$$

such that $\det(D_u f(a)) \neq 0$ then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $u_\varepsilon(t)$ of system (6) such that $u_\varepsilon(0) \to a$ when $\varepsilon \to 0$.

(b) If the real part of all the eigenvalues of $D_u f(a)$ are negative, then the periodic solution $u_\varepsilon(t)$ is stable, if the real part of some eigenvalue of $D_u f(a)$ is positive then the periodic solution is unstable.

3. Outline of the proofs: Paraboloid

The following theorem gives a normal form for polynomial vector fields on the paraboloid.

Theorem 3

If $\mathbb{M}^2$ is a paraboloid, then system (3) *can be written as*

$$\begin{aligned} x' &= \mathcal{G}(x,y,z)A(x,y,z) + E(x,y,z) + 2yF(x,y,z), \\ y' &= \mathcal{G}(x,y,z)B(x,y,z) - D(x,y,z) - 2xF(x,y,z), \\ z' &= \mathcal{G}(x,y,z)C(x,y,z) - 2yD(x,y,z) + 2xE(x,y,z), \end{aligned} \tag{8}$$

with

$$\mathcal{G}(x,y,z) = x^2 + y^2 - z,$$

A, B, C, D, E and F are arbitrary polynomials of $\mathbb{R}[x,y,z]$.

3. Outline of the proofs: Paraboloid (a)

In order to determine the invariant parallels we must consider the intersection of the planes $z = k$, $k > 0$ with $\mathbb{M}^2$.

By Proposition 1 it is necessary that $g(x, y, z) = z - k$ be a factor of the extactic polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X}) = P_3$, where

$$P_3(x, y, z) = -2yD(x, y, z) + 2xE(x, y, z).$$

We have at most $m - 1$ factors of the form $z - k$.

This proves statement (a) of the theorem.

3. Outline of the proofs: Paraboloid (b)

Consider the polynomial vector field (5)

$$\mathcal{X}(x,y,z) = (2y(x-2k)+h(z), -2x(x-2k), 2x\,h(z))\,,$$

where

$$h(z) = \varepsilon z^{m-k-1}\prod_{i=1}^{k}(z-i), \quad \varepsilon > 0 \ small.$$

3. Outline of the proofs: Paraboloid (b)

Thus

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = P_3(x,y,z) = 2xh(z) = 2\varepsilon x z^{m-k-1} \prod_{i=1}^{k}(z-i).$$

This implies that, for $1 \leq k \leq m-1$ fixed, vector field (5) has exactly k invariant parallels given by $z = i$, $i = 1, \ldots, k$.

It is easy to see that this vector field has no equilibria on the invariant parallels.

In order to complete the proof we need to show that these invariant parallels are limit cycles.

3. Outline of the proofs: Paraboloid (b)

Note that the paraboloid $x^2 + y^2 - z = 0$ can be written in the explicit form $z = x^2 + y^2$.

In the coordinates (x, y) vector field (5) has the form

$$\mathcal{X}^*(x, y) = \left(2y(x - 2k) + h^*(x^2 + y^2), -2x(x - 2k)\right),$$

with

$$h^*(x^2 + y^2) = \varepsilon(x^2 + y^2)^{m-k-1} \prod_{i=1}^{k}(x^2 + y^2 - i), \quad \varepsilon > 0 \text{ small}.$$

3. Outline of the proofs: Paraboloid (b)

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ the above vector field is equivalent to

$$\frac{dr}{d\theta} = \frac{\varepsilon r^{2(m-k-1)} \prod_{i=1}^{k}(r^2 - i)\cos\theta}{-2(r\cos\theta - 2k) - \varepsilon r^{2(m-k-1)-1} \prod_{i=1}^{k}(r^2 - i)\sin\theta}.$$

Expanding $dr/d\theta$ in Taylor series with respect to ε at $\varepsilon = 0$ we have

$$\frac{dr}{d\theta} = \frac{r^{2(m-k-1)-1} \prod_{i=1}^{k}(r^2 - i)\cos\theta}{-2(r\cos\theta - 2k)}\,\varepsilon + O(\varepsilon^2). \tag{9}$$

3. Outline of the proofs: Paraboloid (b)

Equation (9) satisfies the hypotheses of the Averaging Theorem.

The averaged function (7) can be written as

3. Outline of the proofs: Paraboloid (b)

$$
\begin{aligned}
f(r) =& \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\cos\theta}{-2(r\cos\theta-2k)}\right)d\theta \\
=& -\frac{1}{4\pi}\left(r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\right)\int_0^{2\pi}\frac{\cos\theta}{(r\cos\theta-2k)}d\theta \\
=& -\frac{1}{4\pi}\left(r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\right)g(r),
\end{aligned}
$$

where

$$
g(r) = 2\pi\left(\frac{\sqrt{4k^2-r^2}-2k}{r\sqrt{4k^2-r^2}}\right).
$$

It is easy to check that $g(r) < 0$ for $0 < r < k$.

Therefore, for $0 < r < k$, the simple zeros of f are given by $r = \sqrt{i}$, for $i = 1, \ldots, k$, which correspond to the k limit cycles of $\mathcal{X}^*$.

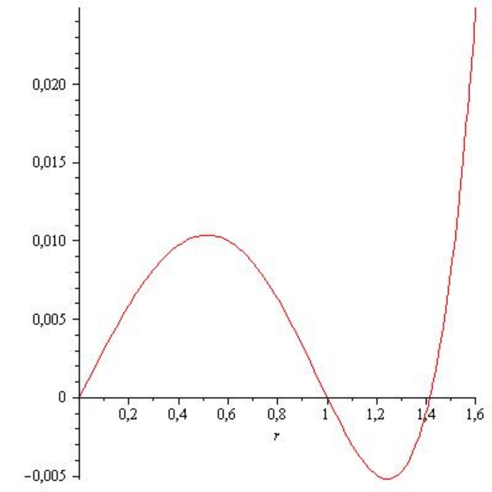
The stability of each limit cycle is easily determined by the sign of the derivative of f at each simple zero.

In short, Theorem 1 is proved.

$\blacksquare$

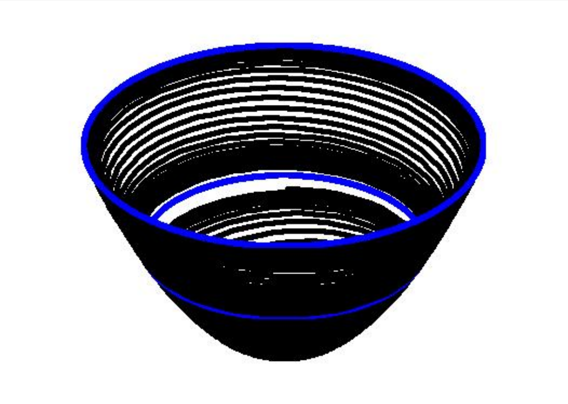
3. Outline of the proofs: Example

Take $m = 3$ and $k = 2$. The graph of the averaged function (7) is given by



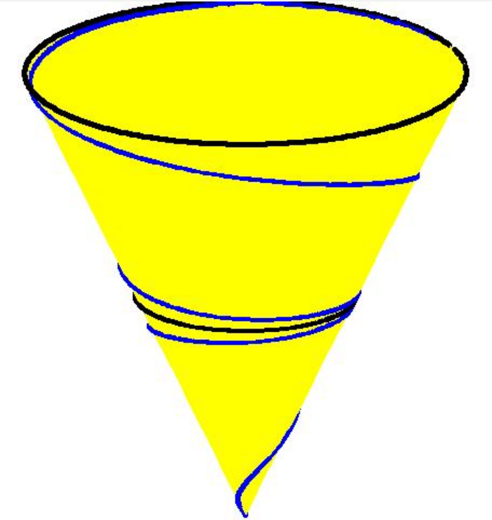
3. Outline of the proofs: Example

Take $m = 3$ and $k = 2$. The phase portrait of the vector field (5) on the paraboloid is depicted in the next figure.



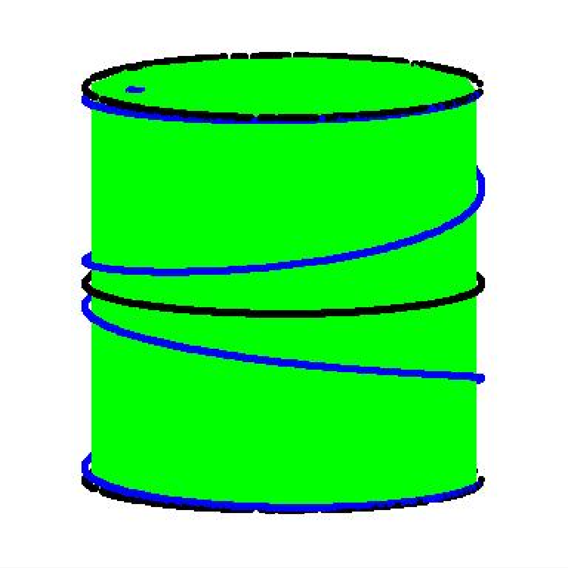
3. Other examples: Cone

Take $m = 3$ and $k = 2$. The phase portrait on the cone.



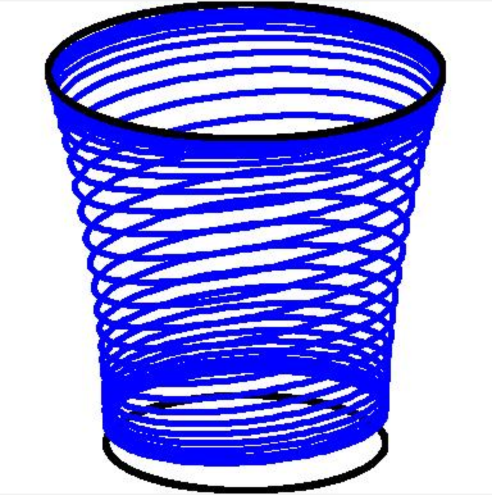
3. Other examples: Cylinder

Take $m = 3$ and $k = 3$. The phase portrait on the cylinder.



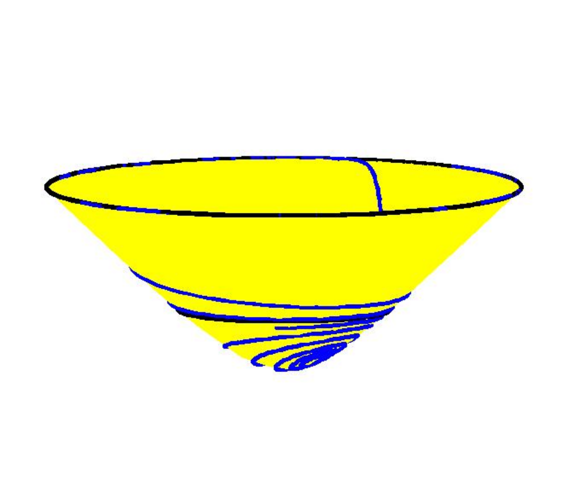
3. Other examples: One–sheet hyperboloid

Take $m = 3$ and $k = 2$. The phase portrait on the one–sheet hyperboloid.



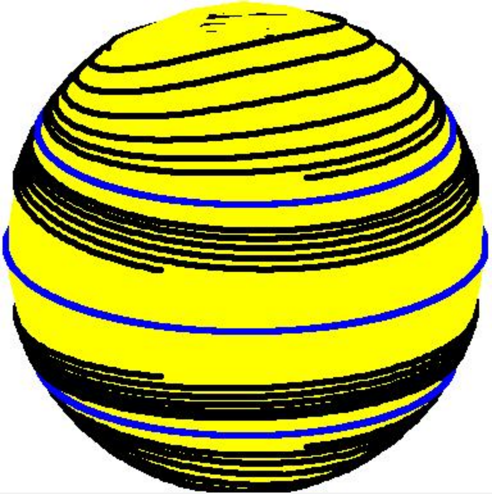
3. Other examples: Two–sheet hyperboloid

Take $m = 3$ and $k = 2$. The phase portrait on the two–sheet hyperboloid.



3. Other examples: Sphere

Take $m = 4$ and $k = 3$. The phase portrait on the sphere.



Plan of the talk:

1. Introduction and generalities;
2. Our results on quadrics of revolution;
3. Outline of the proofs;
4. **Open problem.**

4. Open problem: Liénard systems

Consider the planar polynomial system (Liénard system)

$$x' = y - f(x), \quad y' = -x, \tag{10}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a real polynomial function of degree n.

Here the symbol $[x]$ will denote the integer part function of x.

4. Open problem: Liénard systems

Lins Neto, de Melo and Pugh stated the following conjecture.

A. Lins Neto, W. de Melo, C.C. Pugh, *On Liénard equations*, in: Proc. Symp. Geom. and topol, in: Lectures Notes in Math., vol. 597, Springer–Verlag, 1977, 335–357.

Conjecture 1

The Liénard system (10) *has at most* $[(n-1)/2]$ *limit cycles.*

4. Open problem: Liénard systems

Table : Values of $[(n-1)/2]$ for $1 \leq n \leq 7$.

n	$[(n-1)/2]$
1	0
2	0
3	1
4	1
5	2
6	2
7	3

4. Open problem: Liénard systems

Lins Neto, de Melo and Pugh proved that Conjecture 1 has an affirmative answer for the cases $n = 1$, $n = 2$ and $n = 3$.

A. Lins Neto, W. de Melo, C.C. Pugh, *On Liénard equations*, in: Proc. Symp. Geom. and topol, in: Lectures Notes in Math., vol. 597, Springer–Verlag, 1977, 335–357.

4. Open problem: Liénard systems

Dumortier, Panazzolo and Roussarie shown that the conjecture is not true for $n \geq 7$ providing one additional limit cycle to the ones predicted by the conjecture.

F. DUMORTIER, D. PANAZZOLO, R. ROUSSARIE, *More limit cycles than expected in Liénard equations*, Proc. Amer. Math. Soc., **135** (2007), 1895–1904.

4. Open problem: Liénard systems

In 2011 De Maesschalck and Dumortier proved that the conjecture is not true for $n \geq 6$ providing two additional limit cycles to the ones predicted by the conjecture.

P. De Maesschalck, F. Dumortier, *Classical Liénard equation of degree $n \geq 6$ can have $[(n-1)/2]+2$ limit cycles*, J. Differential Equations, **250** (2011), 2162–2176.

4. Open problem: Liénard systems

In 2012, thirty five years after the statement of the conjecture, it was proved by Li and Llibre that the conjecture holds for $n = 4$.

C. Li, J. Llibre, *Uniqueness of limit cycle for Liénard equations of degree four*, J. Differential Equations, **252** (2012), 3142–3162.

4. Open problem: Liénard systems

Based on the previous comments we have the following problem.

Problem 2

Prove (or disprove) that Liénard systems (10) *of degree* 5 *have at most* 2 *limit cycles.*

THE END.

THANK YOU VERY MUCH.