On Central Configurations of the n–Body Problem

Luis Fernando Mello

Universidade Federal de Itajubá – UNIFEI
E-mail: lfmelo@unifei.edu.br

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Plan of the talk:

1. Introduction: Newtonian n–body problem;
2. Homographic solution and central configuration;
3. Main open problem;
4. Andoyer equations and some results;
5. My last article;
6. Open problem.

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1. Introduction: Newtonian n–body problem

The **Newtonian n–body problem** consists of studying the motion of n punctual bodies with positive masses $m_1, \ldots, m_n$ interacting amongst themselves through no other forces than their mutual gravitational attraction according to Newton's gravitational law.

1. Introduction: Equations of motion

For $i = 1, 2, \ldots, n$, the **equations of motion** are given by

$$m_i \ddot{r}_i = m_i \frac{d^2 r_i}{dt^2} = \sum_{j=1,\, j\neq i}^{n} \frac{G m_i m_j}{|r_i - r_j|^2} \frac{1}{|r_i - r_j|} (r_j - r_i).$$

Here we take $G = 1$ as the gravitational constant;

$r_j \in \mathbb{R}^d$, $d = 2, 3$, is the position vector of the body j;

$r_{ij} = |r_i - r_j|$ is the Euclidean distance between bodies i and j.

So, for $i = 1, 2, \ldots, n$, the equations of motion can be written as

$$\ddot{r}_i = \sum_{j=1,\, j\neq i}^{n} \frac{m_j}{r_{ij}^3} (r_j - r_i), \qquad (1)$$

1. Introduction: Center of mass

The **center of mass** of the system is given by

$$c = \frac{1}{M} \sum_{j=1}^{n} m_j r_j,$$

where $M = m_1 + \ldots + m_n$ is the total mass.

Without loss of generality, we consider the center of mass at the origin of the inertial system.

Such a system is called **inertial barycentric system**.

1. Introduction: Configurations

A **configuration** for the n bodies is a vector

$$r = (r_1, \ldots, r_n) \in \mathbb{R}^{nd} \backslash \Delta,$$

where

$$\Delta = \{r_i = r_j, i \neq j\}$$

is the set of collisions.

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2. Homographic solution

Two configurations for the n bodies

$$r = (r_1, \ldots, r_n), \qquad \bar{r} = (\bar{r}_1, \ldots, \bar{r}_n)$$

are **similar** if we can pass from one configuration to the other by a dilation or a rotation of $\mathbb{R}^d$.

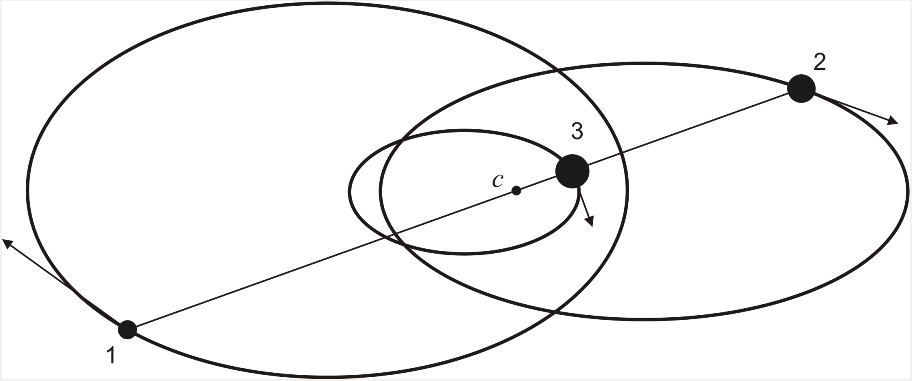
A **homographic solution** of the n–body problem is a solution such that the configuration of the n–bodies at the instant t remains similar to itself as t varies.

2. Euler's homographic solution

The first homographic solution was found in 1767 by Euler in the 3–body problem, for which three bodies are **collinear** at any time.

L. EULER, *De moto rectilineo trium corporum se mutuo attahentium*, Novi Comm. Acad. Sci. Imp. Petrop., **11** (1767), 144–151.

2. Euler's homographic solution



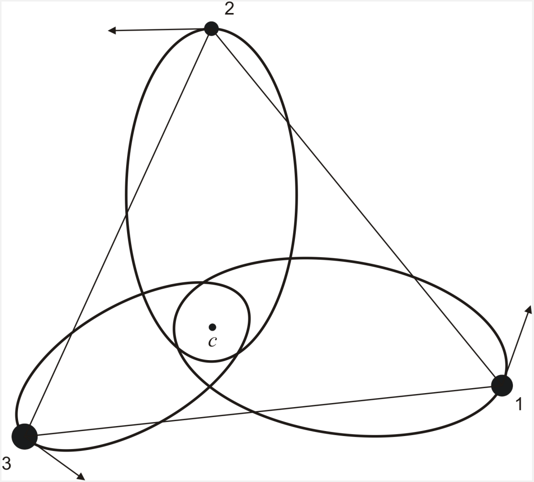
Euler's homographic solution.

2. Lagrange’s homographic solution

In 1772 Lagrange found other homographic solution in the 3–body problem, where the three bodies are at any time at the vertices of an **equilateral triangle**.

J.L. LAGRANGE, *Essai sur le problème de trois corps*, Œuvres, vol 6, Gauthier–Villars, Paris, 1873.

2. Lagrange’s homographic solution



Lagrange’s homographic solution

2. Homographic solution

Pictures in the internet with Euler and Lagrange homographic solutions:

http://www.d.umn.edu/mhampton/threebodies123.gif

http://www.d.umn.edu/mhampton/fourbodycollinear1234.gif

2. Central configuration

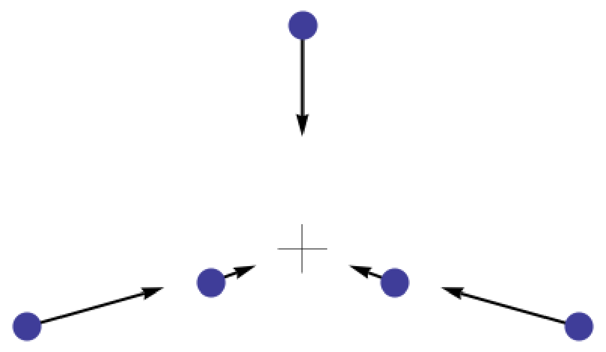
At a given instant $t = t_0$ the configuration of the n bodies is **central** if

$$\ddot{r}_i = \lambda r_i, \qquad \lambda < 0, \tag{2}$$

for all $i = 1, \ldots, n$.

In other words, the acceleration vector of each body is pointing towards the origin (center of mass) with the magnitude of the acceleration proportional to the distance from the origin. See the next figure.

2. Central configuration



Picture from Scholarpedia.

2. Central configuration

It can be proved that, in this case,

$$\lambda = -\frac{U}{2I}, \quad U = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{r_{ij}}, \quad I = \frac{1}{2M} \sum_{i<j} m_i m_j |r_i - r_j|^2,$$

where U is the Newtonian potential and I is the moment of inertia of the system.

2. Central configuration

The previous figure suggests what will happen if the bodies are released from a central configuration with zero initial velocity.

All bodies accelerate toward the origin (center of mass) in such a way that the configuration collapses homothetically.

http://www.scholarpedia.org/w/images/a/a8/Centralconfigurations_lagrangehomothetic.gif

2. Central configuration

From equations (1) and (2) the **equations of the central configurations** can be written as

$$\lambda r_i = \sum_{j=1, j\neq i}^{n} \frac{m_j}{r_{ij}^3}(r_j - r_j), \tag{3}$$

for $i = 1, 2, \ldots, n$.

2. Classes of central configurations

If we have a central configuration, any dilation or any rotation (centered at the center of mass) of it provides another central configuration.

Two central configurations are **related** if we can pass from one to the other through a dilation or a rotation.

This definition gives an equivalence relation in the set of central configurations.

So we can study classes of central configurations defined by the above equivalence relation.

Theorem 1 (Laplace)

The configuration of the n bodies in a homographic solution is central at any instant of time.

A. WINTNER, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, 1941.

2. Converse of Laplace Theorem

There is a converse of Laplace Theorem.

Theorem 2

If we have a central configuration, it is possible to choose velocities for the bodies in order to construct a homographic solution for the n–body problem.

C. Vidal and G. Renildo, *Homographic solutions in the n–body problem*, Cubo, **6** (2004), 185–207.

For example, if the bodies are released from a central configuration with initial velocities normal to their position vectors and with magnitudes proportional to their distances from the origin, then each body will describe an elliptical orbit as in the Kepler problem.

http://www.scholarpedia.org/w/images/0/0f/Centralconfigurations_crookedman.gif

2. Planar central configurations

In this talk we are mainly interested in

planar central configurations,

that is $d = 2$.

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3. Main open problem

The main general open problem for planar central configurations is due to Wintner and Smale:

Problem 1 (Wintner/Smale)

Is the number of classes of planar central configurations finite for any choice of the positive masses $m_1, \dots, m_n$?

S. SMALE, *Mathematical problems for the next century*, Math. Intelligencer, **20** (1998), 7–15.

A. WINTNER, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, 1941.

3. Wintner/Smale's problem

It is easy to prove that there are exactly five classes of central configurations in the 3–body problem (3 due to Euler and 2 due to Lagrange).

Hampton and Moeckel provided an affirmative answer to the problem for $n = 4$ bodies.

M. Hampton and R. Moeckel, *Finiteness of relative equilibria of the four–body problem*, Invent. Math., **163** (2006), 289–312.

3. Wintner/Smale's problem

Alternatively, A. Albouy and V. Kaloshin proved the finiteness when $n = 4$ and, when $n = 5$, the number is finite, except perhaps if the masses belong to a subvariety of the space of positive masses.

A. ALBOUY AND V. KALOSHIN, *Finiteness of central configurations of five bodies in the plane*, Ann. of Math., **176** (2012), 535–588.

The problem remains open for $n > 5$.

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4. Andoyer equations

Consider the **Andoyer equations**

$$f_{ij} = \sum_{\substack{k=1 \\ k \neq i,j}}^{n} m_k \left(R_{ik} - R_{jk} \right) \Delta_{ijk} = 0, \tag{4}$$

for $1 \leq i < j \leq n$, where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$.

Note that Δ_{ijk} is twice the oriented area of the triangle formed by the bodies at r_i, r_j and r_k.

We have $n(n-1)/2$ equations in (4).

4. Andoyer equations

We have the following lemma.

Lemma 1

Consider n bodies with positive masses $m_1, m_2, \ldots, m_n$ and position vectors $r_1, r_2, \ldots, r_n$ in a planar non–collinear configuration. Then

$$\lambda r_i = -\sum_{j \neq i} m_j R_{ij} \,(r_i - r_j),$$

for $1 \leq i \leq n$*, is equivalent to*

$$f_{ij} = \sum_{k \neq i,j} m_k (R_{ik} - R_{jk}) \Delta_{ijk} = 0,$$

for $1 \leq i < j \leq n$*.*

4. Outline of the proof of Lemma 1

Suppose that

$$\lambda r_i = -\sum_{k \neq i} m_k R_{ik}(r_i - r_k)$$

holds for $1 \leq i \leq n$.

Taking $i, j \in \{1, 2, \ldots, n\}$, $j \neq i$, we have

$$\lambda r_i = -\sum_{k \neq i,j} m_k R_{ik}(r_i - r_k) - m_j R_{ij}(r_i - r_j) \tag{5}$$

and

$$\lambda r_j = -\sum_{k \neq i,j} m_k R_{jk}(r_j - r_k) - m_i R_{ji}(r_j - r_i). \tag{6}$$

The difference between (6) and (5) gives

$$\lambda(r_i - r_j) = -\sum_{k \neq i,j} m_k [R_{ik}(r_i - r_k) - R_{jk}(r_j - r_k)] - [m_j R_{ij} - m_i R_{ij}](r_i - r_j). \quad (7)$$

Taking the wedge product by $r_i - r_j$ in both sides of (7) we have

4. Outline of the proof of Lemma 1

$$
\begin{aligned}
0 &= -\sum_{k\neq i,j} m_k[R_{ik}(r_i - r_k)\wedge(r_i - r_j) - R_{jk}(r_j - r_k)\wedge(r_i - r_j)] \\
&= -\sum_{k\neq i,j} m_k[R_{ik}\Delta_{ikj} + R_{jk}\Delta_{jki}] \\
&= -\sum_{k\neq i,j} m_k[-R_{ik}\Delta_{ijk} + R_{jk}\Delta_{ijk}] \\
&= \sum_{k\neq i,j} m_k(R_{ik} - R_{jk})\Delta_{ijk} \\
&= f_{ij}.
\end{aligned}
$$

Therefore, $f_{ij} = 0$ for all $1 \leq i < j \leq n$.

$\blacksquare$

4. Some applications: 1. Lagrange

Consider 3 non–collinear bodies with positive masses.

The 3 Andoyer equations can be written as

$$f_{12} = m_3 \, (R_{13} - R_{23}) \, \Delta_{123} = 0,$$
$$f_{13} = m_2 \, (R_{12} - R_{23}) \, \Delta_{132} = 0,$$
$$f_{23} = m_1 \, (R_{12} - R_{13}) \, \Delta_{123} = 0.$$

As $m_i > 0$ and $\Delta_{ijk} \neq 0$ (the bodies are non–collinear) it follows that $R_{12} = R_{13} = R_{23}$. This implies that $r_{12} = r_{13} = r_{23}$.

In other words, the bodies **with any value of masses** are at the vertices of an equilateral triangle.

So we have the following theorem.

Theorem 3 (Lagrange)

Consider 3 non–collinear bodies with positive masses. These bodies are in a central configuration if and only if they are at the vertices of an equilateral triangle.

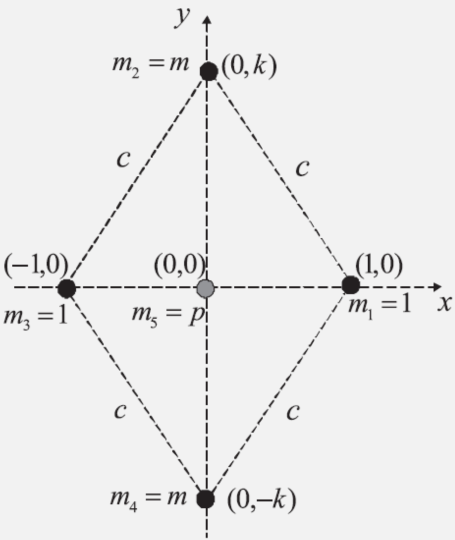
4. Some applications: 2. Roberts

Consider 5 bodies with the masses and positions

$$m_1 = m_3 = 1, \quad m_2 = m_4 = m, \quad m_5 = p,$$

$$r_1 = (1, 0), \quad r_3 = (-1, 0), \quad r_2 = (0, k), \quad r_4 = (0, -k), \quad r_5 = (0, 0),$$

according to the following figure.


y
$m_2 = m$
$(0,k)$
c
c
$(-1,0)$
$(0,0)$
$(1,0)$
$m_3 = 1$
$m_5 = p$
$m_1 = 1$
x
c
c
$m_4 = m$
$(0,-k)$

The 10 Andoyer equations are either trivially satisfied or are equivalent to

$$f_{12} = 0,$$

which can be written as

$$(R_{13} - R_{23})\Delta_{123} + m(R_{14} - R_{24})\Delta_{124} + p(R_{15} - R_{25})\Delta_{125} = 0,$$

or equivalently as

$$m\left(\frac{2}{c^3} - \frac{1}{4k^3}\right) + p\left(1 - \frac{1}{k^3}\right) + \left(\frac{1}{4} - \frac{2}{c^3}\right) = 0.$$

Taking $m = 1$ and $p = -1/4$ the above equation is satisfied for all $k \in \mathbb{R}^+$.

4. Some applications: 2. Roberts

We have the following theorem.

Theorem 4

In the five–body problem for the masses $(1, 1, 1, 1, -1/4)$*, there exists a one–parameter family of central configurations where the four bodies with equal masses are at the vertices of a rhombus with the remaining body located at the center.*

Corollary 1

The number of classes of central configurations in the five–body problem for the masses $(1, 1, 1, 1, -1/4)$ *is* **not** *finite.*

G.E. Roberts, *A continuum of relative equilibria in the five–body problem*, Physica D, **127** (1999), 141–145.

4. Some applications: 2. Roberts

Remark 1

Roberts Theorem does not give a negative answer to Wintner/Smale Problem 1.

Note that in the Wintner/Smale Problem we must have **positive** *masses.*

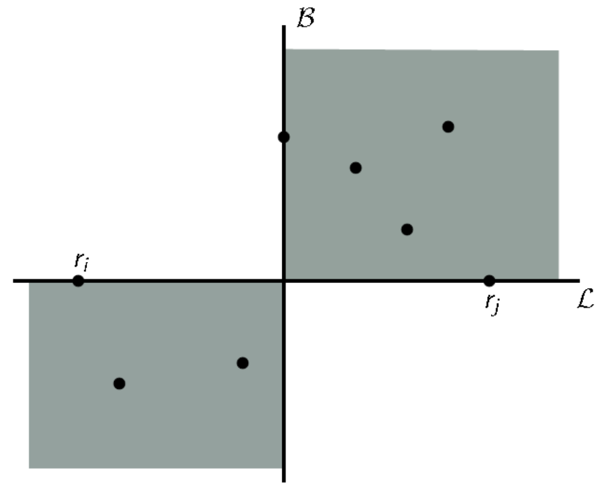
But, it shows that the algebraic equations defining the central configurations admit a kind of **degeneracy***.*

Theorem 5 (Mediatrix theorem)

Let $r = (r_1, \ldots, r_n)$ be a planar central configuration and let r_i and r_j, $i \neq j$, be any two of its positions. Then if one of the two open cones determined by the line $\mathcal{L}$ through r_i and r_j and its mediatrix $\mathcal{B}$ contains points of the configuration, so does the other cone. See the next figure.

R. Moeckel, *On central configurations*, Math. Z. **205** (1990), 499–517.

4. Some applications: 3. Mediatrix theorem



The line $\mathcal{L}$ through r_i and r_j and its mediatrix $\mathcal{B}$. This configuration can not be central since the bodies are in the same open cone.

We have the following two corollaries of Mediatrix Theorem.

Corollary 2

There is no a central configuration for the n–body problem with exactly $n-1$ collinear bodies, for $n > 3$.

Thus, a non–collinear 4–body central configuration is either strictly convex or concave (one body is strictly inside the triangle formed by the other three bodies).

Corollary 3

Suppose a concave configuration of 4 bodies whose triangle has an angle greater than $\pi/2$. Then this configuration is not a central configuration.

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5. My last article

The landmark for the study of convex central configurations in the planar 4–body problem is the article

W.D. MacMillan and W. Bartky, *Permanent configurations in the problem of four bodies*, Trans. Amer. Math. Soc., **34** (1932), 838–875.

Main result.

Theorem 6

For any positive values of m_1, m_2, m_3, m_4 there exists a convex planar central configuration with these masses.

5. My last article

From now on, we study central configurations of the planar 4–body problem that satisfy:

1. The configuration is convex;
2. The configuration possesses two pairs of equal masses;
3. The bodies are numbered 1, 2, 3, 4 in cyclic order.

5. My last article

Perez–Chavela and Santoprete studied the case of equal opposite masses.

Theorem 7

Let $r = (r_1, r_2, r_3, r_4)$ be a convex central configuration with equal opposite masses $m_1 = m_3$ and $m_2 = m_4$. Then the configuration forms a rhombus.

E. Perez–Chavela and M. Santoprete, *Convex four–body central configurations with some equal masses*, Arch. Ration. Mech. Anal., **185** (2007), 481–494.

5. My last article

The following conjecture can be found in

A. ALBOUY, Y. FU, S. SUN, *Symmetry of planar four–body convex central configurations*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **464** (2008), 1355–1365.

Conjecture 1

There is a unique convex planar central configuration having two pairs of equal masses located at the adjacent vertices of the configuration and it is an isosceles trapezoid.

See also

A. ALBOUY, H.E. CABRAL AND A.A. SANTOS, *Some problems on the classical n–body problem*, Celest. Mech. Dyn. Astr., **113** (2012), 369–375.

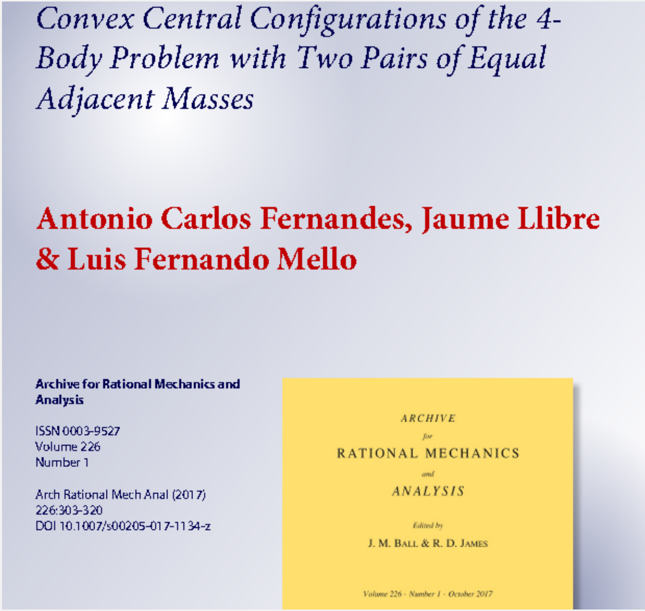
5. My last article

By using the Andoyer equations and many technical results we solve Conjecture 1 proving the following theorem.

Theorem 8 (Fernandes, Llibre and Mello)

Consider a convex configuration of 4 *bodies with position vectors* r_1, r_2, r_3, r_4 *and masses* m_1, m_2, m_3, m_4. *Suppose that* $m_1 = m_2 = \mu$, $m_3 = m_4 = m$, *and* r_1, r_2, r_3 *and* r_4 *are disposed counterclockwise at the vertices of a convex quadrilateral. Then the only possible central configuration performed by these bodies is an isosceles trapezoid.*

5. My last article



Convex Central Configurations of the 4-Body Problem with Two Pairs of Equal Adjacent Masses

Antonio Carlos Fernandes, Jaume Llibre & Luis Fernando Mello

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6. Open problem

Consider $n \geq 3$ bodies in a plane.

Problem 2 (Co–circular central configurations)

Prove (or disprove) that in the planar n–body problem the equal mass regular n–gon is the **only** *central configuration with all bodies on a common circle and with its center of mass at the center of the circle.*

6. Open problem

M. HAMPTOM, *Co–circular central configurations in the four–body problem*, in: Dumortier, F., Henk, B. et al. (eds.) Equadiff 2003, Proceedings of the International Conference on Differential Equations, pp. 993–998. World Scientific Publishing Co., Singapore (2005).

J. LLIBRE AND C. VALLS, *The co–circular central configurations of the 5–body problem*, J. Dynam. Differential Equations, **27** (2015), 55–67.

Thank you very much for your attention.