Volume Renormalization for Singular Yamabe Metrics in Higher Codimension

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Overview

1. CCE Metrics
2. Boundary Singular Yamabe Metrics
3. Singular Yamabe Metrics

CCE Metrics

CCE Metrics

Suppose M^{n+1} is a compact manifold with boundary $\partial M = \Sigma^n$.

A **defining function** for Σ in M is a smooth function r on M such that

$$\begin{aligned} r > 0 &\quad \text{in} \quad \mathring{M}, \\ r = 0 &\quad \text{on} \quad \Sigma, \\ dr \neq 0 &\quad \text{on} \quad \Sigma. \end{aligned}$$

A Riemannian metric g^+ on $\mathring{M}$ is called **conformally compact** if $(M, r^2 g^+)$ is a compact Riemannian manifold with boundary.

Let $g = r^2 g^+$ and $g|_\Sigma = h$.

CCE Metrics

The conformally compact manifold $(\mathring{M}, g^+)$ induces a well defined conformal structure $[h]$ on Σ.

$(\Sigma, [h])$ is called the **conformal infinity** of the conformally compact manifold $(\mathring{M}, g^+)$.

A conformally compact metric with

$$\mathrm{Ric}(g^+) = -ng^+$$

is called a **conformally compact Einstein** (CCE) metric.

CCE Metrics

Theorem (Graham-Lee '91, Graham '00)

Let $(\mathring{M}, g^+)$ *be a CCE manifold with conformal infinity* $(\Sigma, [h])$*. Then*

1. *For any* $h \in [h]$*, there is a unique defining function* r *such that*

$$|dr|_g = 1$$

   in a neighborhood $[0, \delta) \times \Sigma$ *of* Σ*, and* $r^2 g^+|_\Sigma = h$*.*

2. *With this defining function,* g^+ *can be written as*

$$g^+ = \frac{1}{r^2}(dr^2 + h_r),$$

   where h *is a 1-parameter family of metrics with parameter* r*. Also,* h *expands in* r *as*

$$h_r = h + r^2 h_2 + \cdots \textit{(even powers)} \cdots r^{n-1} h_{n-1} r^n h_n + o(r^n).$$

CCE Metrics

The volume of $(\mathring{M}, g^+)$ is infinite.
One considers the volume of the region $\{r > \epsilon\}$ in M to find:

Theorem (Henningson-Skenderis '98, Graham '00)

1. *The volume of $\{r > \epsilon\}$ in g^+ expands as*

$$\mathrm{Vol}_{g^+}\{r > \epsilon\} = c_0\epsilon^{-n} + c_1\epsilon^{-n+1} + \cdots + c_{n-1}\epsilon^{-1} + \mathcal{E}\log\left(\frac{1}{\epsilon}\right) + V + o(1),$$

   where $c_j = 0$ for each odd j.

2. *If n is odd, $\mathcal{E}$ is zero and V is independent of $h \in [h]$; and if n is even, $\mathcal{E}$ is independent of $h \in [h]$.*

We call V the **renormalized volume** of $(\mathring{M}, g^+)$,
and call $\mathcal{E}$ the **energy** of Σ in M.

Boundary Singular Yamabe Metrics

Boundary Singular Yamabe Metrics

Let (M^{n+1}, g) be a compact Riemannian manifold with boundary $\partial M = \Sigma^n$. Write $g|_\Sigma = h$.

A complete metric g^+ on $\mathring{M}$ is called *the* **boundary singular Yamabe** (∂SY) metric for (M, g) if
there is a function u with

$$\begin{aligned} u > 0 \quad &\text{on} \quad \mathring{M}, \\ u = 0 \quad &\text{on} \quad \Sigma, \end{aligned}$$

such that $g^+ = u^{-2} g$ has

$$R_{g^+} = -n(n+1).$$

The scalar curvature condition can be seen as a PDE in u:

$$L[u] := 1 - |du|_g^2 + \frac{2}{n+1} u \Delta_g u + \frac{1}{n(n+1)} u^2 R_g = 0.$$

Boundary Singular Yamabe Metrics

Theorem (Aviles-McOwen '88)

The problem

$$L[u] = 0 \text{ on } \mathring{M}, \qquad u|_\Sigma = 0$$

has a unique solution.

Under the identification of a neighborhood of Σ in M with $[0, \delta) \times \Sigma$, g takes the form

$$g = dr^2 + h_r$$

where r is the distance to Σ w.r.t. g, and h_r is a 1-parameter family of metrics on Σ with parameter r such that $h_0 = h$.

The solution u expands in r as

$$u \sim r + u_2 r^2 + \cdots + u_{n+1} r^{n+1} + \mathcal{L} r^{n+2} \log r + u_{n+2} r^{n+2} o(r^{n+2}),$$

where $\mathcal{L}$ and u_m for $m < n+2$ are locally formally determined, and u_{n+2} is formally undetermined.

Boundary Singular Yamabe Metrics

The volume form for $(\mathring{M}, g^+)$ is

$$dV_{g^+} = u^{-n-1} dV_g .$$

Again, the volume is infinite.

The volume of the region $\{r > \epsilon\}$ is

$$\mathrm{Vol}_{g^+}\{r > \epsilon\} = c_{-n}\epsilon^{-n} + c_{-n+1}\epsilon^{-n+1} + \cdots + c_{-1}\epsilon^{-1} + \mathcal{E}\log(\frac{1}{\epsilon}) + V + o(1).$$

Theorem (Graham '17)

$\mathcal{E}$ is independent of choice of $g \in [g]$.

Generically, V is not a conformal invariant in this setting.

Singular Yamabe Metrics

Singular Yamabe Metrics

Let (M^{n+k}, g) be a closed Riemannian manifold.
Suppose Σ^n is an embedded submanifold of M.
Write $g|_\Sigma = h$.

A complete metric g^+ on $M\backslash\Sigma$ is called a **singular Yamabe** (SY) metric for (M, Σ, g) if there is a function u with

$$\begin{aligned} u > 0 &\quad \text{on} \quad M\backslash\Sigma, \\ u = 0 &\quad \text{on} \quad \Sigma, \end{aligned}$$

such that $g^+ = u^{-2}g$ has

$$R_{g^+} = -(n-k+2)(n+k-1).$$

Singular Yamabe Metrics

Writing the scalar curvature condition as

$$2L[u] = (n+2-k) - (n+k)|du|_g^2 + 2u\Delta_g u + \frac{1}{n+k-1} R_g u^2 = 0,$$

we have that

Theorem (Aviles-McOwen '88)

The problem

$$L[u] = 0 \quad \text{on } M \backslash \Sigma,$$
$$u|_\Sigma = 0$$

has a unique solution if and only if $k < n+2$.

[Mazzeo '91, Mazzeo-Pacard '96] When $k \geq n+2$, a solution need not exist, need not be unique if it does exist, and need not be polyhomogeneous.

Singular Yamabe Metrics

Assume $k < n+2$ throughout.
Let t be the g-distance to Σ.

Theorem (Mazzeo '91)

The solution u to the singular Yamabe problem expands in t as

$$u \sim t + u_2 t^2 + \cdots + u_{n+1} t^{n+1} + \mathcal{L} t^{n+2} \log t + u_{n+2} t^{n+2} + o(t^{n+2}),$$

where the function $\mathcal{L}$, and each u_ℓ for $\ell < n+2$, is a smooth function on $\mathbb{S}^{k-1} \times \Sigma$ formally determined by g. u_{n+2} is formally undetermined.

Theorem (K-McKeown)

The solution to the singular Yamabe problem u of the form $u = t\bar{v} + O(t^{n+2} \log t)$, with $\bar{v}|_\Sigma = 1$, and $\bar{v} \in C^\infty(M)$.

Singular Yamabe Metrics

We may identify a neighborhood of Σ in M with $[0,\delta) \times \mathbb{S}^{k-1} \times \Sigma$.
Any point q in this neighborhood may be written as (t, ω, p).
For $p \in \Sigma$, define

$$\mathcal{Y}_j(p) = \begin{cases} \mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_j & j \text{ even}, \\ \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus \cdots \oplus \mathcal{H}_j & j \text{ odd}, \end{cases}$$

where $\mathcal{H}_j$ is the space of eigenfunctions of $\mathbb{S}^{k-1}$ corresponding to the eigenvalue $-j(j+k-2)$.

Any smooth function f on M in a neighborhood of Σ can be written as $f(t, \omega, p)$ on $[0,\delta)_t \times \mathbb{S}^{k-1} \times \Sigma^n$.

In these coordinates, any smooth f expands in powers of t locally as

$$f = f_0 + tf_1 + \cdots + t^\ell f_\ell + \cdots ,$$

with f_ℓ smooth on $\mathbb{S}^{k-1} \times \Sigma$, and $f_\ell(-, p) \in \mathcal{Y}_\ell(p)$ for each ℓ.

When ℓ is odd, f_ℓ integrates to zero over $\mathbb{S}^{k-1}$.

Singular Yamabe Metrics

For a smooth function f written as

$$f = f_0 + tf_1 + \cdots + t^\ell f_\ell + \cdots ,$$

we may think of the coefficients of odd powers of t as odd functions on the $(k-1)$-sphere and the coefficients of even powers of t as even functions on the $(k-1)$-sphere under the antipodal action.

In particular, at each $p \in \Sigma$, the coefficient of t^ℓ in the expansion of the solution to the singular Yamabe problem u satisfies

$$u_\ell(-, p) \in \mathcal{Y}_{\ell-1}(p) \quad \text{for } \ell < n+2.$$

Also, $\mathcal{L}(-, p) \in \mathcal{Y}_1(p)$.

With

$$u \sim t + u_2 t^2 + \cdots + u_{n+1} t^{n+1} + \mathcal{L} t^{n+2} \log t + u_{n+2} t^{n+2} + o(t^{n+2}),$$

the volume form of $g^+ = u^{-2} g$ has the form

$$dV_{g^+} = \vartheta dt dV_h dV_{\mathring{b}},$$

where

$$\vartheta = t^{-n-1} \left(\vartheta_0 + \vartheta_1 t + \cdots + \vartheta_n t^n + o(t^n) \right)$$

is such that for each $p \in \Sigma$, $\vartheta_\ell \in \mathcal{Y}_\ell(p)$.

So, ϑ_ℓ for odd ℓ all integrate to zero over $\mathbb{S}^{k-1}$.

Singular Yamabe Metrics

The volume of $(M\backslash\Sigma, g^+)$ is infinite.
Considering the region $\{t > \epsilon\}$, we have

Theorem (K-McKeown)

The volume of the region $\{t > \epsilon\}$ expands as

$$\mathrm{Vol}_{g^+}\{t > \epsilon\} = c_0\epsilon^{-n} + c_1\epsilon^{-n+1} + \cdots + c_{n-1}\epsilon^{-1} + \mathcal{E}_{n,k}\log\left(\frac{1}{\epsilon}\right) + V_{n,k} + o(1),$$

where $c_j = 0$ for each odd j.
If n is odd, $\mathcal{E}_{n,k}$ is zero and $V_{n,k}$ is independent of $g \in [g]$.
If n is even, $\mathcal{E}_{n,k}$ is independent of $g \in [g]$.

$\mathcal{E}_{n,k}$ is a conformal invariant in terms of local Riemannian invariants.
$V_{n,k}$ is an absolute conformal invariant.

Observe the reintroduction of parity to the invariance!

Examples

For $n = 1$ and $k = 2$, if $M = \mathbb{S}^3$, $V_{1,2}$ is a conformal invariant associated with knot embeddings.

Suppose $(M, g) = (\mathbb{S}^{n+k}, \mathring{g})$, and let $\Sigma = \mathbb{S}^n$ be an equatorial sphere.

We have

$$\mathcal{E}_{n,k} = (-1)^{\frac{n}{2}} \frac{4\pi^{\frac{n+k}{2}}}{\left(\frac{n}{2}\right)!\Gamma\left(\frac{k}{2}\right)} \text{for } n \text{ even,}$$

$$V_{n,k} = (-1)^{\frac{n+1}{2}} \frac{2\pi^{1+\frac{n+k}{2}}}{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{k}{2}\right)} \text{for } n \text{ odd.}$$

In particular, $\mathcal{E}_{2,2} = -4\pi^2$ and $V_{1,2} = -4\pi^2$.
The latter is the renormalized volume associated with the equatorial unknot in $\mathbb{S}^3$.

Examples

Let (M^{2+k}, g) be a closed Riemannian manifold with $1 \leq k \leq 3$.
Suppose Σ^2 is a closed surface embedded in M.

We have

$$\mathcal{E}_{2,k} = \frac{\mathrm{Vol}_{\mathring{b}}(\mathbb{S}^{k-1})}{8(4-k)} \int_\Sigma \left(k(|\mathfrak{H}|^2 + 4\,\mathrm{tr}_h(P|_{T\Sigma})) + 4|\mathring{\mathfrak{L}}|^2 - 8R_h \right) dV_h.$$

where
$h = g|_{T\Sigma}$,
$\mathring{b}$ is the standard metric on $\mathbb{S}^{k-1}$,
P is the Schouten tensor,
$\mathfrak{L}$ is the second fundamental form of Σ in M, and
$\mathfrak{H}$ is its mean curvature.

When $k = 1$, this is

$$\mathcal{E}_{2,1} = \frac{1}{2} \int_\Sigma \left(|\mathring{\mathfrak{L}}|^2 - R_h \right) dV_h$$

Variational Formulae

Recall that

$$u \sim t + u_2 t^2 + \cdots + u_{n+1} t^{n+1} + \mathcal{L} t^{n+2} \log t + u_{n+2} t^{n+2} + o(t^{n+2}).$$

The *log*-coefficient $\mathcal{L}$ in u is the restriction to the unit normal bundle of a *linear* function on the entire normal bundle, i.e., of a one-form. Also, $\mathcal{L}$ is a conformal invariant of weight $-n$.

The formally undetermined term u_{n+2} is *not* the restriction to the normal bundle of a linear function.
Still, when n is odd, it has a conformally invariant linear *part* (not locally determined), which thus determines a conformally invariant one-form of weight $-n$ on $N\Sigma$:

$$(u_{n+2})_p(X) = \frac{k}{\mathrm{Vol}_{\mathring{b}}(\mathbb{S}^{k-1})} \int_{SN_p\Sigma} u_{n+2}(Y) \langle X, Y \rangle dV_{\mathring{b}}(Y),$$

where $SN_p\Sigma$ is the unit normal sphere at $p \in \Sigma$.

Variational Formulae

Theorem (K-McKeown)

If n is odd and $k > 1$, then $\mathcal{L} = 0$.

Let $\mathcal{F} : (-\delta, \delta) \times \Sigma \hookrightarrow M$ be a smooth variation of Σ with $\mathcal{F}(0, \cdot) = \mathrm{id}_\Sigma$. Suppose $X = \frac{d}{ds}\mathcal{F}(s, \cdot)|_{s=0} \in \Gamma(\Sigma, N\Sigma)$ is its variation field. For each s, let $\Sigma_s = \mathcal{F}(s, \Sigma)$, and let $\mathcal{E}_{n,k}(s)$ and $V_{n,k}(s)$ be the energy and volume corresponding to (M, Σ_s).

If n is even, then $$\frac{d}{ds}\Big|_{s=0}\mathcal{E}_{n,k}(s) = C_{n,k}\int_\Sigma \mathcal{L}(X)dV_h.$$

If n is odd, then $$\frac{d}{ds}\Big|_{s=0}V_{n,k}(s) = C_{n,k}\int_\Sigma u_{n+2}(X)dV_h.$$

Here $C_{n,k} = (-1)^n\frac{(n+k)(n^2+n+2k-4)}{k(n-k+2)}\mathrm{Vol}_{\mathring{b}}(\mathbb{S}^{k-1})$.

Variational Formulae

For $(M, g) = (\mathbb{S}^{n+k}, \mathring{g})$, the equatorial spheres $\Sigma = \mathbb{S}^n$ are critical.

Singular Yamabe Metrics

As mentioned, when $k \geq n+2$, a solution need not exist, need not be unique if it does exist, and need not be polyhomogeneous.

We may still construct *formal* solutions to the singular Yamabe problem, with desired properties that allow us to run the renormalization process beyond $k < n+2$.

For each n, there are finitely many exceptional codimensions k for which this process does not yield ideal (but still interesting) results.
Call this set S_n.

All the mentioned results of [K-McKeown] still hold for $k \notin S_n$.

The results pertaining to $\mathcal{E}_{n,k}$ still hold geometric meaning, as it is an integral of Riemannian invariants.
The results about $V_{n,k}$ for $k \geq n+2$ *do not*, since $V_{n,k}$ is generally not a canonical quantity anymore.

Thank You.

Singular Yamabe Metrics

Set
$P_n(p) = n + 2 + 2np - 4p^2$ and
$Q_n(p) = 2n + 1 + 2(n-2)p - 4p^2$.

If n is even, define

$$E_n = \left\{ P_n(p) : p = 0, \cdots, \left\lfloor \frac{n}{4} \right\rfloor \right\}$$
$$O_n = \left\{ Q_n(p) : p = 0, \cdots, \left\lfloor \frac{n}{4} \right\rfloor \right\}.$$

If n is odd, define

$$E_n = \left\{ P_n(p) : p = 1, \cdots, \left\lfloor \frac{n}{2} \right\rfloor \right\}$$
$$O_n = E_n \cup \{n+2\}.$$

Let S_n denote the set $E_n \cup O_n$.

Singular Yamabe Metrics

For $k \notin S_n$, the previous results for $\mathcal{E}_{n,k}$ and $V_{n,k}$ hold as they are.
Assume $k \in S_n$.

Theorem (K-McKeown)

If $k \in O_n$, then there exists u with the expansion

$$u \sim t + u_2 t^2 + \cdots + u_{n+1} t^{n+1} + \mathcal{L} t^{n+2} \log t + u_{n+2} t^{n+2} + o(t^{n+2})$$

*satisfying $R_{u^{-2}g} = -(n+2-k)(n+k-1) + O(t^{n+1})$.
This u is unique given the additional constraint that, for each positive root ν to $Q_n(\nu) = k$, the coefficient $u_{\nu+1}$ integrates to zero on each fiber of the normal sphere bundle to Σ; this is equivalent to requiring that $\frac{u}{t}$ be smooth on M, and is thus a conformally invariant condition.*

The previous results for $\mathcal{E}_{n,k}$ and $V_{n,k}$ thus hold for $k \in O_n$ too.

Singular Yamabe Metrics

Theorem (K-McKeown)

Suppose that $k \in E_n$. Let ν be the smallest positive root to $P_n(\nu) = k$.

1. *If $\nu \neq \frac{n}{2}$, a formal expansion u exists of the form*
$$u = t + t^2 u_2 + \cdots + t^\nu u_\nu + t^{\nu+1}(\log t)A + t^{\nu+1}u_{\nu+1},$$
satisfying $R_{u^{-2}g} = (k-n-2)(n+k-1) + o(t^{\nu+1})$.
Such u is unique to order $o(t^{\nu+1})$ subject to the requirement that the integral of $u_{\nu+1}$ over each fiber of the normal sphere bundle to Σ vanish. This is not a conformally invariant condition.

2. *If $\nu = \frac{n}{2}$, u has a formal expansion of the form*
$$u = t + t^2 u_2 + \cdots + t^{\frac{n}{2}} u_{\frac{n}{2}} + t^{\frac{n}{2}+1}(\log t)^2 A + t^{\frac{n}{2}+1} u_{\frac{n}{2}+1},$$
satisfying $R_{u^{-2}g} = (k-n-2)(n+k-1) + o(t^{\frac{n}{2}+1})$. Uniqueness is the same as in the preceding case.

In either case, A is a pointwise conformal invariant of weight $-\nu$.