Introduction to Arithmetic Quantum Chaos

(Igor Prokhorenkov)

(really arithmetic quantum chaos)

Many fields come together in this subject: number theory, quantum physics, spectral geometry, graph theory, etc.

In quantum physics, we are interested in the energy levels E_i of a quantum mechanical system.

$$E_1 \leq E_2 \leq \ldots$$

are the eigenvalues of some Hamiltonian operator:

$$H\psi_j = E_j\psi_j$$

The quantum Hamiltonian H looks like

$$H\psi = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} + V(x,y)\psi$$

in $\mathbb{R}^2$, where V is a potential (which could be matrix-valued). Unfortunately, unless the system is very simple like a harmonic oscillator, it is almost impossible to compute the levels explicitly. Instead of studying each E_j individually, we want to study statistical properties of the E_j's.

Generally speaking, there are two ways to study the statistical properties. One way is to use trace formulas. For example:

A circle of radius 1, which is the same as $\mathbb{R}/2\pi$. $H = -\frac{\partial^2}{\partial\theta^2}$. To solve the problem

$$H\psi_j = \lambda_j\psi_j,$$

we get $\sin(j\theta)$, $\cos(j\theta)$, $j = 0, 1, 2, \ldots$; $\lambda_j = j^2$. Then the trace formula is

$$\frac{1}{2\pi}\sum_{n\in Z} e^{-n^2t} = \frac{1}{2(\pi t)^{1/2}} + \frac{1}{(\pi t)^{1/2}}\sum_{m=1}^{\infty}\exp\left(-\frac{\pi^2m^2}{t}\right),\ t > 0.$$

This is an example of the Poisson summation formula. On the left side, this is

$$\frac{1}{2\pi}\sum e^{-\lambda_j t},$$

which comes entirely from the spectrum. On the right side, you have

$$\exp\left(-\frac{\pi^2m^2}{t}\right) = \exp\left(-\frac{(2\pi m)^2}{4t}\right),$$

and $2\pi m$ is the length of a closed geodesic. So trace formulas are like this: a function of the spectrum is a sum of geometric quantities, usually involving geodesics. One can think of this as relating classical to quantum mechanics, or spectral theory and geometry. A name for this is a Tauberian Theorem. Important in the above trace formula is that the right side has clear asymptotic behavior as $t \to 0$.

Igor showed us that there are many numerical distributions coming from different areas. Random numbers from the Poisson distribution have spacings similar to prime spacings, and eigenvalue spacings for hyperbolic surfaces. Radiating isotopes and the Sinai billiard problem have similar distributions.

Quantum chaos: how can you tell by looking at a quantum system or if the underlying classical system is completely integrable or if it is chaotic. Wigner had an idea that one should model quantum models with statistics of eigenvalues of really large symmetric matrices. If you draw a histogram of eigenvalues of 200 random 50x50 matrices, the distribution is called semicircular. In number theory this histogram also appears. The next attempt: study spacings between eigenvalues. A famous meeting of Dyson (quantum mechanics) and Montgomery (zeros of zeta function): it turns out that the spacing distributions are similar.