LOCAL CLASS GROUPS

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All rings considered here are commutative with identity 1.

1. Local rings and localization

Definition 1.1. A ring R is *local* if it has a unique maximal ideal $\mathfrak{m}$, emphasized by the notation $(R, \mathfrak{m})$. If $p \subset R$ is a prime ideal, then $S = R - p$ is a multiplicatively closed set and we may form $R_p = S^{-1}R = \{\frac{a}{s} : a \in A, s \in S\}$ addition and multiplication making sense because we can find common denominators. This process is called *localization at* p because pR_p is the unique maximal ideal in R_p.

Example 1.2. (a) Any field k is a local ring, since the only proper ideal (0) is maximal.

(b) If A is an integral domain, then $S = A - \{0\}$ is multiplicatively closed (A has no zero-divisors) and $S^{-1}A = K(A)$ is the standard construction of the fraction field of A.

(c) The integers $\mathbb{Z}$ do not form a local ring, but for any prime ideal p we find a local ring in $\mathbb{Z}_p = \{\dfrac{f}{g} : f, g \in \mathbb{Z}, g \notin p\}$.

(d) $\mathbb{C}[x]$ is not local, but the ideal $(x-2) \subset \mathbb{C}[x]$ is maximal so

$$\mathbb{C}[x]_{(x-2)} = \{\frac{f(x)}{g(x))} : f(x), g(x) \in \mathbb{C}[x], g(2) \neq 0\}$$

is a local ring with maximal ideal generated by $(x-2)$.

Example 1.3. The term *local* comes from geometry. Let p be a point on a complex manifold M with open neighborhood U. One can form the ring of functions

$$\mathcal{O}_M(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic on } U\}$$

If U is sufficiently small, then $\mathfrak{m}_p = \{f : f(p) = 0\} \subset \mathcal{O}_M(U)$ is a maximal ideal. The *ring of germs of holomorphic functions at* p is

$$\mathcal{O}_{M,p} = \varprojlim_{p \in U} \mathcal{O}_M(U)$$

formed by equivalence classes of pairs (U, f) as above, where $(U, f) = (V, g)$ if $f = g$ on $U \cap V$. The ring $\mathcal{O}_{M,p}$ is a local with maximal ideal $\mathfrak{m}_p$ being formed by the functions f which vanish at p. For small neighborhoods U, we have a natural inclusion $\mathcal{O}_M(U)_{\mathfrak{m}_p} \subset \mathcal{O}_{M,p}$ which is not onto because one can find a sequence of holomorphic functions with radius of convergence tending to zero.

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Example 1.4. If $X \subset \mathbb{C}^n$ is a complex algebraic variety defined by ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ and $U \subset X$ is a Zariski open neighborhood of $p \in X$, one can similarly consider the ring

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} : f \text{ is locally a rational function}\}$$

of regular functions on U. For the open set $U = X$, $\mathcal{O}_X(X) \cong A(X) = \mathbb{C}[x_1, \dots x_n]/I$ is the affine coordinate ring of X. Again we can form a ring of germs of functions at p by taking the inverse limit $\mathcal{O}_{X,p} = \varprojlim \mathcal{O}_X(U)$ and due to the polynomial nature of the functions we have an isomorphism

$$\mathcal{O}_{X,p} \cong \mathcal{O}_X(X)_{\mathfrak{m}_p} \cong (\mathbb{C}[x_1, \dots, x_n]/I)_{\mathfrak{m}_p}$$

showing that the ring of germs of polynomial functions on X is the localization of the ring of polynomial functions on X. This explains the language in the definition.

Remark 1.5. The local rings in the last two examples give quite different global information. The local ring $\mathcal{O}_{M,p}$ on the complex manifold M tells only $\dim M$ since it is determined by the holorphic functions on a disk. The local ring $\mathcal{O}_{X,p}$ on the algebraic variety X defined by a prime ideal I actually determines the birational equivalence class of X, since $K(X) = K(\mathbb{C}[x_1, \dots, x_n]/I) = K(\mathcal{O}_{X,p})$ can be recovered (the local ring is obtained by inverting polynomials outside $\mathfrak{m}_p$ and the function field is obtained by inverting all non-zero polynomials). The birational equivalence class of an algebraic variety X is entirely determined by its function field $K(X)$ as a $\mathbb{C}$-algebra and $\dim X$ is the transcendence degree of $K(X)$ over $\mathbb{C}$.

2. Power series rings and completion

Definition 2.1. The *completion* of a local ring $(R, \mathfrak{m})$ is the inverse limit $\widehat{R} = \varprojlim R/\mathfrak{m}^n$ formed by sequences $\{a_n\}$ with $a_n \in R/\mathfrak{m}^n$ such that a_n is the restriction of a_{n+1} (the sequence is coherent). The ring $\widehat{R}$ is also a local ring with maximal ideal $\widehat{\mathfrak{m}}$ and there is a natural (flat) homomorphism $(R, \mathfrak{m}) \to (\widehat{R}, \widehat{\mathfrak{m}})$ of local rings sending $a \in R$ to the constant sequence $a_n = a$. The ring R is *complete* if this map is an isomorphism.

Example 2.2. Let $R = \mathbb{C}[x_1, \dots x_n]$ and complete at the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Then $R/\mathfrak{m} \cong \mathbb{C}$ and in general $R/\mathfrak{m}^n$ appears as polynomials truncated at degree n, giving the inverse system

$$\mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_n) \leftarrow \mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_n)^2 \leftarrow \mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_n)^3 \leftarrow$$

from which one sees that the coherent sequences correspond to power series, built up one term at a time, thus $\widehat{R} \cong \mathbb{C}[[x_1, \dots x_n]]$.

Definition 2.3. Local rings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are *analytically isomorphic* if $\hat{A} \cong \hat{B}$.

Example 2.4. Let $x \in X$ be a smooth point on a complex algebraic variety of dimension n. Then x is a **smooth** point of X if the local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is a **regular local ring**, meaning that $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = n$. In this case $\widehat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[x_1, \dots, x_n]]$, so analytic isomorphism only determines $\dim X$, much like the situation of Example 1.3 above for germs of holomorphic functions on a complex manifold. From this we see that analytic isomorphism can only be useful in studying singularities of algebraic varieties. Considering the local ring of germs of holomorphic functions at x as well we have

$$\mathcal{O}_{X,x} \subset \mathcal{O}^{\mathrm{Hol}}_{X,x} \subset \widehat{\mathcal{O}}_{X,x}$$

Because the germs of holomorphic functions are locally given as power series with positive radius of convergence.

Example 2.5. Consider the nodal cubic curve $X \subset \mathbb{C}^2$ given by equation $y^2 = x^2 - x^3$. Then X is singular only at the origin. In variables $X = y - x, Y = y + x$ the equation becomes $XY + f_3(X, Y)$ where $f_3(X, Y) = (Y - X)^3/8$ is a homogeneous cubic polynomial and the analytic isomorphism class is given by completing the ring $\mathbb{C}[X, Y]/(XY + f_3(X, Y))$ to obtain $\mathbb{C}[[X, Y]]/(XY + f_3(X, Y))$.

Working in the power series ring we can make an analytic change of coordinates to obtain the ring $\mathbb{C}[[x', y']]/(x'y')$. Inductively set $(X_0, Y_0) = (X, Y)$. Since $f_3(X, Y)$ is a homogeneous cubic in X, Y, we can find quadratic h_1, g_1 so that $f_3(X, Y) = Xh_1 + Y g_1$. Setting $X_1 = X_0 + g_1, Y_1 = Y + 0 + h_1$ we have $X_1Y_1 = X_0Y_0 + f_3(X_0, Y_0) + h_1g_1$ so that

$$X_0Y_0 + f_3(X_0, Y_0) = X_1Y_1 + f_4(X_1, Y_1).$$

Continuing in this way, we find a sequence $(X_0, Y_0), (X_1, Y_1), \ldots$ which converges in $\mathbb{C}[X, Y]$ to (x', y') when the equation takes the form $x'y' = 0$. Here the completion of an integral domain is not even an integral domain thanks to the zero-divisors x' and y'.

3. Normal rings

Definition 3.1. Let A be an integral domain with fraction field K. The *integral closure* (or *normalization*) $\overline{A}$ of A is the set elements $x \in K$ which satisfy a monic polynomial equation $p(x) = 0$ with coefficients in A. The ring A is *normal* if $A = \overline{A}$.

Proposition 3.2. *A is normal $\iff$ A_p is normal for each prime ideal $p \subset A$.*

Example 3.3. (a) The integers $\mathbb{Z}$ form a normal ring, for if $a/b \in \mathbb{Q}$ satisfies an equation $x^r + c_1x^{r-1} + \cdots + c_r = 0$ with $c_i \in \mathbb{Z}$ with $(a, b) = 1$, multiplying by b^r gives

$$a^r + c_1a^{r-1}b + \ldots c_rb^r = 0$$

so that $b|a^r$ and it follows that $b = 1$.

(b) The ring $R = \mathbb{Z}[\sqrt{5}]$ is not normal, because $x = (1 \pm \sqrt{5})/2 \in \mathbb{Q}(\sqrt{5})$ are roots of the quadratic monic equation $x^2 - x - 1$. The integral closure is $\overline{R} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. In general $\mathbb{Z}[\sqrt{d}]$ is normal exactly for square-free d which are not equivalent to 1 mod 4.

This definition has geometric consequences. If $X \subset \mathbb{C}^n$ is an algebraic variety defined by the ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, then we say X is **normal** if each local ring $\mathcal{O}_{X,x}$ is normal, which by Proposition 3.2 is equivalent to normality of the affine coordinate $A(X) = \mathbb{C}[x_1, \ldots, x_n]/I$. This definition implies that the singular locus of X has codimension ≥ 2 among other things: hence a normal curve is smooth and a normal surface has only finitely many singularities.

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a regular local ring. Then R is a normal ring.*

Geometrically this tells us that every smooth variety is normal.

4. Class groups

From now on I will only work with **noetherian** rings, meaning that every ideal is finitely generated. The class of noetherian rings include fields (there is only one ideal, namely (0)!) and PIDs (ideals are generated by one element). If R is noetherian, then so are $R[x]$, any localization $S^{-1}R$, and any quotient ring R/I. In particular, if $X \subset \mathbb{C}^n$ is an algebraic variety, then the affine coordinate ring $A(X) = \mathbb{C}[x_1, \dots, x_n]/I_X$ and all the local rings $\mathcal{O}_{X,x}$ are noetherian.

We've all seen ideals in a ring: there are similar structures in the fraction field.

Definition 4.1. A **fractional ideal** J of an integral domain R is an R-submodule $J \subset K = K(R)$ for which there exists $d \in R$ with $dJ \subset R$.

Remark 4.2. Since $I = dJ \subset R$ forms an ideal in the usual sense, the fractional ideals are simply $J = \dfrac{1}{d} I \subset K$ where $I \subset R$ is a traditional ideal.

We would like to be able to make a group out the fractional ideals with identity element $(1) = R \subset K$. We can multiply fractional ideals, but existence of inverses is a problem. Given $I \subset K$, the natural candidate for I^{-1} is $(I : R) = \{a \in K : aI \subset R\}$ and one would hope that $(I^{-1})^{-1} = I$. With the notation $\overline{I} = ((I : R) : R)$, we say that I is **divisorial** if $I = \overline{I}$.

Proposition 4.3. *$\overline{I}$ is the smallest divisorial ideal containing I.*

With this in mind, the set $D(R)$ of divisorial ideals is a multiplicative monoid under the product $I \cdot J = \overline{IJ}$. This still doesn't produce a group, but we're almost there.

Theorem 4.4. *$D(R)$ is a group if and only if R is a normal ring (integrally closed). In this case $D(R)$ is freely generated by the height 1 primes in R.*

A **principal fractional ideal** has the form $(a) = \{a \cdot r : r \in R\} \subset K$ for $0 \neq a \in K$. The principal fractional ideals form a subgroup of $D(R)$ denoted $P(R)$.

Definition 4.5. The **class group** of R is the quotient group $\mathrm{Cl}(R) = D(R)/P(R)$.

Example 4.6. With Remark 4.2 in mind, we see that if R is a PID, then every fractional ideal is principle. Indeed, if $dJ \subset R$ is equal to (c), then $J = (c/d) \subset K$. Thus if R is a PID, then $\mathrm{Cl}\, R = 0$. The converse is true if R is a Dedekind domain (equivalently every fractional ideal is divisorial).

Example 4.7. The classic example of a non-UFD is $R = \mathbb{Z}[\sqrt{-5}]$ due to the non-unique factorization $2 \cdot 3 = (1 - \sqrt{-5}) \cdot (1 + \sqrt{-5})$. One can show that $\mathrm{Cl}\, R \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the non-principal ideal $J = (2, 1 + \sqrt{-5})$.

In fact, this seems to be the first occurrence of the class group. Gauss worked with expressions that we now recognize as fractional ideals (the definition of ideal didn't come out until Kummer's work in the later 1800s!) when he was working on unique factorization in the rings $\mathbb{Z}[\sqrt{d}]$. For $d < 0$, he conjectured which such rings have class number 1, which was finally proven by Stark in the 1960s. The problem is still open for $d > 0$.

Theorem 4.8. *A normal integral domain R is a UFD if and only if* $\operatorname{Cl} R = 0$.

Remark 4.9. Geometrically if $X \subset \mathbb{C}^n$ is a normal affine variety, we can define the class group of X to be $\operatorname{Cl} X = \operatorname{Cl} A(X) = \operatorname{Cl} \mathbb{C}[x_1, \dots x_n]/I_X$ (the definition is slightly more involved for projective varieties). When X is *smooth*, then $\operatorname{Cl} X = \operatorname{Pic} X$ is isomorphic to the Picard group of line bundles modulo linear equivalence with tensor product as group operation. When X is singular there is an exact sequence relating the two which takes the form

$$0 \to \operatorname{Pic} X \to \operatorname{Cl} X \to \bigoplus_{p \in \operatorname{Sing} X} \operatorname{Cl} \mathcal{O}_{X,p} \tag{1}$$

We have used this to compute Picard groups of singular surfaces. The hard part is that the **Local Class Groups** $\operatorname{Cl} \mathcal{O}_{X,p}$ are usually very difficult to compute. On the other hand, for any normal local ring $(R, \mathfrak{m})$ there is an injective group homomorphism $\operatorname{Cl} R \hookrightarrow \operatorname{Cl} \widehat{R}$, so we may work in an appropriate power series ring with more tools.

Example 4.10. In the study of surface singularities a prominent role has been played by the rational double points, originally defined by Artin in the early 60s. They have been a test case for various conjectures because their resolutions are well-understood along with other invariants. They are given by the analytic isomorphism classes given in the following table.

Type	Local Equation in $\mathbb{C}[[x, y, z]]$	Class group
$\mathbf{A}_n$	$xy - z^{n+1}$	$\mathbb{Z}/(n+1)\mathbb{Z}$
$\mathbf{D}_n$	$x^2 + y^2z + z^{n-1}$	$\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2/Z$
$\mathbf{E}_6$	$x^2 + y^3 + z^4$	$\mathbb{Z}/3\mathbb{Z}$
$\mathbf{E}_7$	$x^2 + y^3 + yz^3$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbf{E}_8$	$x^2 + y^3 + z^5$	0

5. Some open questions

Srinivas proved that rational double point singularities are completions of UFDs en route to his calculation of $K_0(R) = \mathbb{Z}$ for any complete rational double point ring R [19]. This led him to ask more general questions about normal local integral domains.

Question 5.1. Given a complete normal local ring B, what are the possible images $\operatorname{Cl} A \hookrightarrow \operatorname{Cl} \hat{A} = \operatorname{Cl} B$ over all local rings A with completion isomorphic to B?

Question 5.2. Special case of the above question: when is B the completion of a UFD which is of essentially finite type over $\mathbb{C}$?

Heitmann has given a complete characterization of normal local rings which are completions of UFDs [10], but his constructions are set-theoretic and rarely produce geometric rings. Srinivas and Parameswaran proved that any isolated local complete intersection singularity is analytically isomorphic to a UFD [17]. For hypersurface singularities, we are able to achieve the same result of arbitrary singularities:

Theorem 5.3. *Let $A = \mathbb{C}[[x_1, \dots x_n]]/f$, where f is a polynomial defining a variety V which is normal at the origin. Then there exists a hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ and a point $p \in X$ such that $R = \mathcal{O}_{X,p}$ is a UFD and $\widehat{R} \cong A$.*

Regarding Question 5.1, Mohan Kumar [15] proved that for rational double points on a rational surface, the analytic isomorphism class already determines the algebraic isomorphism class (there are 3 exceptions) and in particular Srinivas' question has a unique answer there. In stark contrast to his result, we prove that in general ALL subgroups are possible for rational double point singularities:

Theorem 5.4. *Fix $T \in \{\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8\}$ and a subgroup H of the class group of the completed local ring for a singularity of type T. Then there exists a surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ and a rational double point $p \in S$ of type T such that $\operatorname{Cl} \mathcal{O}_{S,p} \subset \widehat{\operatorname{Cl} \mathcal{O}_{S,p}}$ realizes H.*

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