Introduction to Algebraic Geometry

(Scott Nollet)

The main things algebraic geometers studies are zero sets of polynomials.

Affine varieties: Let $\mathbf{k}$ be a field. (examples are $\mathbf{k} = \mathbb{R}, \mathbb{C}, \mathbb{Z}/p\mathbb{Z}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{F}_p$) We define affine n-space as

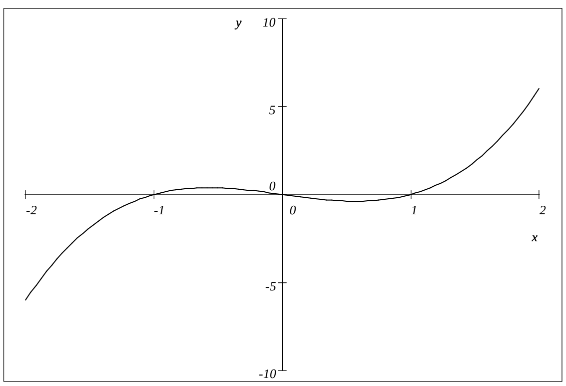
$$\mathbb{A}^n_{\mathbf{k}} = \{\bar{a} = (a_1, \ldots, a_n \in \mathbf{k}^n)\}$$

Let $\{f_\alpha\}$ be a collection of polynomials in $\mathbf{k}[x_1, \ldots, x_n]$. Let

$$Z(\{f_\alpha\}) = \left\{\bar{a} \in \mathbb{A}^n_{\mathbf{k}} : f_\alpha(\bar{a}) = 0 \text{ for all } \alpha\right\}.$$

Examples:

- $Z(y - x^3 + x)$, $\mathbf{k} = \mathbb{R}$:



- $\mathbf{k} = \mathbb{Q}$, $Z(1 - x^2 - y^2) = \left\{\left(\frac{a}{c}, \frac{b}{c}\right) : a, b, c \in \mathbb{Z}, a^2 + b^2 = c^2\right\}$.
- $\mathbf{k} = \mathbb{Q}$, $Z(1 - x^p - y^q)$ — algebraic number theory, Fermat's last theorem
- $\mathbf{k} = \mathbb{Z}_2$, $\mathbb{A}^{10}_{\mathbf{k}} = \{$10-bit binary numbers$\}$. If take a finite field extension $\mathbb{F}_{2^{10}}$ – get applications in computer science.

Another concrete example: take $Z(y - x, y - x^2, y - x^3, \ldots)$, $\mathbf{k} = \mathbb{R}$ $Z = \{(0,0), (1,1)\}$

Remark If f_α are polynomials, let

$$I = \left\{\sum_{finite} p_\alpha(\bar{x}) f_\alpha(\bar{x}) : p_\alpha \in \mathbf{k}[\bar{x}]\right\}.$$

Claim: $Z(\{f_\alpha\}) = Z(I)$. The set I is an ideal in $\mathbf{k}[\bar{x}]$

Theorem (Hilbert basis theorem, 1899) Every ideal in $\mathbf{k}[\bar{x}]$ is finitely generated. In other words, there exist $p_1, \ldots, p_r$ such that

$$I = (g_1, \ldots, g_r) = \left\{\sum_{j=1}^{r} p_j(\bar{x}) g_j(\bar{x}) : p_j \in \mathbf{k}[\bar{x}]\right\}.$$

For example, $(y - x, y - x^2, y - x^3, \ldots) = (y - x, x - x^2)$

Note that $Z(x^{20}, y) = Z(x, y)$.

Projective varieties: Projective n-dimensional space is

$$\mathbb{P}^n_{\mathbf{k}} = \left\{\text{"lines" through origin in } \mathbb{A}^{n+1}_{\mathbf{k}}\right\}$$
$$= \{t(a_0,\ldots,a_n) : t \in \mathbf{k}, (a_0,\ldots,a_n) \neq (0,\ldots,0)\}$$
$$= \mathbb{A}^{n+1} \setminus \{(0,\ldots,0)\}/\sim$$

where $(a_0,\ldots,a_n) \sim (b_0,\ldots,b_n)$ if there exists $\lambda \neq 0$ in $\mathbf{k}$ such that $\lambda(a_0,\ldots,a_n) = (b_0,\ldots,b_n)$.

Note that $\mathbb{P}^2_{\mathbb{R}}$ is the unit sphere mod the antipodal map. This is not orientable. Note that $S^2 \to \mathbb{P}^2_{\mathbb{R}}$ is a $2-1$ covering map.

On the other hand, $\mathbb{P}^2_{\mathbb{C}}$ is the set of complex lines in $\mathbb{C}^2$.

Note that $\mathbb{P}^1_{\mathbb{R}}$. is $(1,1)$ a zero for $y - x^2$? The answer is no, because $2 - 2^2 \neq 0$. So it is hard to find zeros. To fix this problem,

Definition $f \in \mathbf{k}(x_0,\ldots,x_n)$ is **homogeneous of degree** d if $f(x)$ is the sum of monomials of degree d.

A **projective variety** is a zero set of a set of homogeneous polynomials. The point is that in this case that

$$f(a_0,\ldots,a_n) = 0$$

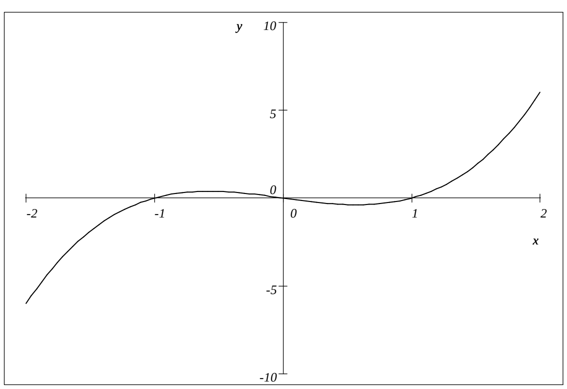
if and only if

$$f(\lambda a_0,\ldots,\lambda a_n) = \lambda^d f(a_0,\ldots,a_n) = 0.$$

One important example of this is :

Example $Z(x_0) \subset \mathbb{P}^n_{\mathbf{k}}$ is the set $\{(a_0,\ldots,a_n) : a_0 = 0\}$, which can be identified with $\mathbb{P}^{n-1}_{\mathbf{k}}$.
What is left over is a copy of $\mathbb{A}^n$,
i.e.$\{(a_0 \neq 0, a_1,\ldots,a_n) : a_j \in \mathbf{k}\} = \{(1, \frac{a_1}{a_0},\ldots,\frac{a_n}{a_0}) : a_j \in \mathbf{k}\} = \{(1,a_1,\ldots,a_n) : a_j \in$
So $\mathbb{P}^n_{\mathbf{k}} = \mathbb{P}^{n-1}_{\mathbf{k}} \cup \mathbb{A}^n$.

Example $Z(y - x^3 + x)$ in $\mathbb{A}^2_{\mathbb{R}}$



Example $Z(y - x^3 + x)$ in $\mathbb{A}^3_{\mathbb{R}}$ (previous picture cross $\mathbb{R}$)

Example $Z(yz^2 - x^3 + xz^2)$ in $\mathbb{A}^3_{\mathbb{R}}$

$yz^2 - x^3 + xz^2 = 0$ makes a cone.

$z = \frac{x^3}{y+x}$

Example $Z(yz^2 - x^3 + xz^2)$ in $\mathbb{P}^2_R$: What happens at infinity? Think of this as $\mathbb{A}^2_R \cup \mathbb{P}^1_R$. At infinity, this is $z = 0$, so $x^3 = 0$. Actually the ends of the curve are meeting at a single point. The cube power means that it meets infinity tangently. Switching coordinates, the equation looks like $z^2 - x^3 + xz^3 = 0$ in $\mathbb{A}^2_R$. So over the real numbers this is a smooth curve, but in complex projective space you get a singularity.

Recall: $\mathbb{A}^n_{\mathbf{k}} = \mathbf{k}^n$, $Z(\{f_i\}) = Z(I)$, $f_i \in \mathbf{k}[x_1, \ldots, x_n]$
$\mathbb{P}^n_{\mathbf{k}} = \left\{\mathbf{k}^{n+1} - \{0\}\right\} / \sim$, $(a_0, \ldots, a_n) \sim \lambda(a_0, \ldots, a_n)$

Back to example: $y = x^3 - x$. How do lines intersect the graph? Could be one point, 3 points, 1 pt+double point, triple point ,

multiplicity of intersection: double point has multiplicity 2, etc.

$y = mx + b$ and $y = x^3 - x$ intersection yields $0 = x^3 - (m+1)x - b$.

If we work over $\mathbb{C}$, we always have three roots with multiplicity. But we still have $x =$constant, which have only one solution, even over $\mathbb{C}$.

To fix: work in $\mathbb{P}^2_{\mathbb{C}}$, and we always get three points of intersection:

$yz^2 - x^3 - xz^2 = 0$, line is $ax + by + cz = 0$. We can always solve this system (eg if $c \neq 0$, $z = \ldots$), and we get $AX^3 + BX^2Y + CXY^2 + DY^3 = 0$.

If $A \neq 0$, then $Y = 0$ can't be a solution. Then get $Y^3\left(A(\frac{X}{Y})^3 + B(\frac{X}{Y})^2 + C(\frac{X}{Y}) + D\right) = 0$, so get 3 solutions for $\frac{X}{Y}$ over $\mathbb{C}$, and thus get 3 points in $\mathbb{P}^3_{\mathbb{C}}$. On the other hand, if $A = 0$, $B \neq 0$, then get two additional solutions, etc. So counting multiplicity, we always get 3 solutions. If $A = B = C = 0$. Then $Y = 0$ is a triple point.

Proposition If $X \subset \mathbb{P}^2_{\mathbf{k}}$, $X = Z(f)$, $\mathbf{k}$ algebraically closed, f homogeneous of degree d. A line $L \subset \mathbb{P}^2_{\mathbf{k}}$ that is not contained in X satisfies $L \cap X = d$ points with multiplicities.

Theorem (Bezout's Theorem) If $X = Z(f) \subset \mathbb{P}^2_{\mathbf{k}}$, $deg(f) = d$; $deg(g) = e$; $Y = Z(g) \subset\subset \mathbb{P}^2_{\mathbf{k}}$, f, g relatively prime, then $X \cap Y = \left\{de \text{ points}\right\}$ (with multiplicity).

What is multiplicity of some multiple intersection point? For instance $y = 0$ intersects with $y = x^3$ at the origin in a triple point. The idea is that $\mathbf{k}[x,y]/(y, y - x^3) \cong \mathbf{k}[x,y]/(y, x^3) \cong \mathbf{k}[x]/(x^3)$ has dimension 3. This idea almost succeeds. Another example: $f = y$, $g = x^2 - x$ Then $\mathbf{k}[x,y]/(y, x^2 - x)$ is not a field, but $\mathbf{k}[x]/(x^2 - x)$ has rank 2. So the dimension gives a count of the sum of multiplicities.

If you want the multiplicity just at the origin, the trick is:

$$\mathbf{k}[x,y]/(y, x^2 - x) = \mathbf{k}[x,y]/(y, x(x-1))$$

Replace $\mathbf{k}[x,y]$ with $\mathbf{k}[x,y]_{(x,y)} = \left\{\frac{f}{g} : f \in \mathbf{k}[x,y], g \in (x,y)\right\}$, the rational functions. Then

$$\text{multiplicity} = \dim_{\mathbf{k}} \frac{\mathbf{k}[x,y]_{(x,y)}}{(y, x(x-1))}.$$

Another reason that $\mathbb{P}^n_{\mathbf{k}}$ is good to work with. Let $X \subset \mathbb{P}^n_{\mathbf{k}}$ be the zero set of an ideal I. The **Hilbert polynomial** is

$$H_{\mathbf{k}}(m) = \dim_{\mathbf{k}} \left(\frac{\mathbf{k}[x_0, \ldots, x_n]}{I}\right)_m,$$

where m means the homogeneous degree m piece. Interesting fact:

Theorem (Hilbert) There exists a polynomial $P(z) \in \mathbb{Q}[z]$ such that $H_X(m) = P(m)$ for $m >> 0$. (P is called the Hilbert polynomial of X)

Example: if $I = (0)$, $X = \mathbb{P}^2_{\mathbf{k}}$. We have $\mathbf{k}[x_0, \ldots, x_2]_0 = k$ constants; $\mathbf{k}[x_0, \ldots, x_2]_1$ has $\dim 3$. $\dim_{\mathbf{k}}\left(\frac{\mathbf{k}[x_0,\ldots,x_n]}{I}\right)_2 = 6$, $\dim_{\mathbf{k}}\left(\frac{\mathbf{k}[x_0,\ldots,x_n]}{I}\right)_3 = 10$. So $P(z) = \frac{(z+1)(z+2)}{2}$.

Geometrically, these Hilbert polynomials are good for computing invariants:

$\dim_{\mathbb{C}}(X) = deg(P_X(z)) = r$

$deg(X) = \left(\text{leading coefficient}\right) r!$

Interpretation: suppose that $X \subset \mathbb{P}^3_{\mathbf{k}}$ is a curve. Then $deg(X) =$the number of points in $X \cap H$, where H is given by one linear equation.

If $\dim(X) = 1$, then $P_X(0) - 1 = \dim(\Omega_C)$, the dimension of holomorphic differential forms.