BIRATIONALITY OF ÉTALE MAPS VIA SURGERY

Abstract. We use a counting argument and surgery theory to show that if $D \subset \mathbb{C}^n$ is a sufficiently general hypersurface, then any local diffeomorphism $F : X \to \mathbb{C}^n$ of simply connected manifolds which is a d-sheet covering away from D has degree $d = 1$ or ∞. In particular, if F is an algebraic map of varieties, then F is birational.

A. Kulikov's Question

The Jacobian Conjecture (Keller, 1939). *If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial biholomorphism (= étale map), then F is injective (= birational).*

Observations:

(1) Several faulty proofs - even recent - see BCW 1982 survey paper.

(2) The image $F(\mathbb{C}^n)$ is Zariski open in $\mathbb{C}^n$ and intersects *every* hypersurface $H \subset \mathbb{C}^n$. Indeed, if $h(\overline{x}) = 0$ is the equation of h, then $D(h \circ F) = Dh \circ F \cdot DF \neq 0$ because h is non-constant, hence $h \circ F$ is non-constant, meaning that the equation $h \circ F = 0$ has solutions in $\mathbb{C}^n$. Therefore $\operatorname{codim} \mathbb{C}^n - F(\mathbb{C}^n) \geq 2$.

(3) There is a hypersurface $D \subset \mathbb{C}^n$ for which the restriction $\mathbb{C}^n - F^{-1}(D) \to \mathbb{C}^n - D$ is a d-sheet covering map for some $d \geq 1$. In fact, the image of $\pi_1(\mathbb{C}^n - F^{-1}(D)) \to \pi_1(\mathbb{C}^n - D)$ is generated by loops which wrap around D with linking number ± 1, called *geometric generators.*

A Generalization of Jacobian Conjecture (Kulikov, 1993). *If $F : X \to \mathbb{C}^n$ is étale with X simply connected and* $\operatorname{codim} \mathbb{C}^n - F(X) \geq 2$, *must $d = 1$?*

Applying the Lefschetz hyperplane theorem, he arrives at the equivalent statement:

Question. *If $D \subset \mathbb{C}^2$ is a curve and $G \subset \pi_1(\mathbb{C}^2 - D)$ is a subgroup of finite index generated by geometric generators, must $G = \pi_1(\mathbb{C}^2 - D)$?*

Example. *Kulikov answers negatively with the following: Let $D \subset \mathbb{P}^2$ be a quartic with 3 cusps - this is given by Zariski as the smallest degree plane curve for which $\mathbb{P}^2 - D$ is non-Abelian. Take a line L at infinity meeting D transversely. Then by Nori's work on Zariski's conjecture, one has an exact sequence*

$$1 \to K \to \pi_1(\mathbb{C}^2 - D) \to \pi_1(\mathbb{P}^2 - D) \to 1$$

in which $K \cong \mathbb{Z}$ is central, generated by a single loop about L. $\pi_1(\mathbb{P}^2 - D)$ is non-Abelian of order 12 $= < a, b : a^2 = b^2, a^4 = 1, (ab)^3 = a^2 >$ and lifting a to $\overline{a}$, $G =< \overline{a} >$ has index 3.

B. Our Results

In view of Kulikov's example, there are still questions: If $D \subset \mathbb{C}^n$ is a hypersurface, X simply connected and $F : X \to \mathbb{C}^n$ étale and a d-sheet cover away from D, when is $d > 1$ possible? What if D is smooth? What if $\pi_1(\mathbb{C}^n - D)$ is Abelian? What happens for general D? Thus we ask the question:

Question. *If X is connected and simply connected, $F : X \to \mathbb{C}^n$ which is a degree d covering map away from a hypersurface $D \subset \mathbb{C}^n$, when must $d = 1$?*

Toy Example:. *Suppose $D \subset \mathbb{C}$ is finite, $H_1(X, \mathbb{Z}) = 0$ and the local diffeomorphism $F : X \to \mathbb{C}$ is a d-sheet cover away from D. Then $d = 1$ or $d = \infty$.*

Notice that both $d = 1$ and $d = \infty$ occur with $X = \mathbb{C}$ and F the identity map or the complex exponential map.

We sketch the proof in the special case. If $D = \{p_1, p_2, \ldots, p_n\}$, choose disjoint open rays l_i emanating from each p_i and set $A = \bigcup l_i$. Then $A \subset \mathbb{C}$ is a 1-submanifold with boundary $\partial A = D$ and $F^{-1}(A) \subset X$ is also a 1-submanifold with connected components C_i.

Lemma 1. *If no C_i is closed in X, then $d = 1$.*

The point is that $X - F^{-1}(\overline{A})$ is path-connected. Choose $a, b \in X - F^{-1}(\overline{A})$ and let τ be a path from a to b in X. After wiggling τ, we may assume that τ misses $F^{-1}(D)$ and meets $F^{-1}(A)$ transversely. If τ meets C_i, pick a point $p \in \overline{C_i} - C_i$. Because F is a local diffeomorphism near p and $F(p) \in D$, we can move the path around the corner, connecting a and b in $X - F^{-1}(\overline{A})$. Now since $\mathbb{C} - \overline{A}$ is contractible, the degree of the covering must be $d = 1$.

Lemma 2. *If some C_i is closed in X, then $d = \infty$.*

Suppose C_1 closed and $F(C_1) = \overline{l_1}$. Because $C_1 \subset X$ is a closed codimension one submanifold and $H_1(X, \mathbb{Z}) = 0$, C_1 disconnects X: i.e. $X - C_1 = U_1 \cup U_2$ disjointly. Both $U_i - F^{-1}(\overline{A}) \to \mathbb{C} - \overline{A}$ are covering maps of degree d_i with $d_1 + d_2 = d$.

Pick $p \in A$, B a small disk about p. A cuts B into two pieces B^+ and B^-. Look at the pieces of $F^{-1}(B) = \bigcup B_i$: the first r are cut in two by C_1, $r + 1$ up to q lie in U_1 and $q + 1$ up to d lie in U_2. So $p \in B^+$ has exactly $r + q$ preimages in U_1 which $p \in B^-$ has exactly q preimages in U_1. This contradicts the even covering property, as the size of the fibres should be constant.

D. Higher Dimensions

The higher dimensional analog of the toy example is the following:

Theorem 1 (Nollet and Xavier, 2007). *Let $F : X \to \mathbb{R}^n$ be a local diffeomorphism, a d-sheet cover away from $D \subset \mathbb{R}^n$, $H_1(X, \mathbb{Z}) = 0$. Suppose that $D = \partial A$, where $A \subset \mathbb{R}^n$ is a real codimension one submanifold with $\mathbb{R}^n - \overline{A}$ simply connected. Then $d = 1, 2$ or ∞ (and $d \neq 2$ if A is oriented).*

Remarks:. *1. In fact, $d \neq 2$ if A is oriented.*

2. The case $d = 2$ actually occurs for an example with $D \subset \mathbb{R}^4$ a Klein bottle.

3. Clearly $H_1(X, \mathbb{Z}) = 0$ necessary, otherwise take $z \mapsto z^d$ from $\mathbb{C} - 0$ to $\mathbb{C}$.

How does one produce such a bounding submanifold A to apply Theorem 1? First we observe a result of Verdier:

Theorem (Verdier, Inventiones 1976). *There is a finite set $S \subset \mathbb{C}$ such that $\mathbb{C}^n - h^{-1}(S) \to \mathbb{C} - S$ is a locally trivial fibration.*

We will say that D is *non-bifurcated* if $0 \notin S$, in other words if D the fibration is trivial near D. With this, our main result is

Theorem 1 (NX 2007). *If D is smooth, connected, non-bifurcated and $F : X \to \mathbb{C}^n$ is a local diffeomorphism which is a d-sheet cover away from D and X is simply connected, then $d = 1$ or $d = \infty$.*

How to produce the submanifold A to apply Theorem 2 towards Theorem 1? Given $D : h = 0$, it's easy to produce a bounding manifold for D: let l be a ray from 0 in $\mathbb{C}$ and set $A = h^{-1}(l)$ to get a real codimension one oriented submanifold of X. It is unlikely that $\mathbb{C}^n - \overline{A}$ will be simply connected. Here we use surgery to replace A with B so that $D = \partial A = \partial B$ and $\mathbb{C}^n - \overline{B}$ IS simply connected.

References

[BCW] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. A.M.S. **7** (1982), 287–330.