Residue currents of coherent sheaves via superconnections

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Contents

The topological space $\Omega_c^{p,q}(X)$

- Let X be an n-dimensional complex manifold.
- Let $\Omega_c^{p,q}(X)$ be the space of compactly supported (p,q)-forms on X, equipped with the usual topology given as follows:
- We say $\omega_n \to \omega$ if for any coordinate chart U and any multi-index α we have

$$||\partial^\alpha \omega_n - \partial^\alpha \omega|| \to 0$$

on U.

Review of currents

- A (p,q)-*current* on X is a continuous linear map from $\Omega_c^{n-p,n-q}(X)$ to $\mathbb{C}$.
- We denote the set of (p,q)-currents on X by $\mathcal{D}^{p,q}(X)$.
- Examples of currents:
  - If $\omega = \sum_{|I|=p,|J|=q} f_{IJ} dz^I \wedge d\bar{z}^J$ where f_{IJ} is a *locally integrable* function on X, then
  $$\omega(\theta) := \int_X \omega \wedge \theta, \ \forall \theta \in \Omega_c^{n-p,n-q}(X)$$
  defines a (p,q)-current on X.
  - If Z is a subvariety of codimension p, then
  $$[Z](\theta) := \int_Z \theta, \ \forall \theta \in \Omega_c^{n-p,n-p}(X)$$
  defines a (p,p)-current on X.

New currents from old ones

- We can extend operations on differential forms to operations on currents by *duality*.
- Let $T \in \mathcal{D}^{p,q}(X)$ be a (p,q)-current on X. We can define a $(p,q+1)$-current $\bar{\partial}T$ as
$$\bar{\partial}T(\theta) := (-1)^{p+q+1}T(\bar{\partial}\theta),\ \forall\theta \in \Omega_c^{n-p,n-q-1}(X).$$
We can define ∂T in a similar way.
- It is compatible with the $\bar{\partial}$-operation on $\Omega^{\bullet,\bullet}(X)$ because for $\omega \in \Omega^{p,q}(X)$ considered as a current as before, we have
$$0 = \int_X \bar{\partial}(\omega \wedge \theta) = \int_X (\bar{\partial}\omega) \wedge \theta + (-1)^{p+q} \int_X \omega \wedge \bar{\partial}\theta.$$

New currents from old ones (cont'd)

- For a $\omega \in \Omega^{s,t}(X)$ and $T \in \mathcal{D}^{p,q}(X)$, we can define a current $\omega \wedge T \in \mathcal{D}^{p+s,q+t}(X)$ as
$$(\omega \wedge T)(\theta) := (-1)^{(s+t)(p+q)} T(\omega \wedge \theta), \ \forall \theta \in \Omega_c^{n-p-s,n-q-t}(X).$$
We can define $T \wedge \omega$ in a similar way.
- In general we *cannot* define the wedge product of two currents.
- We have an inclusion of cochain complexes $(\Omega^{\bullet,\bullet}(X), \bar{\partial}) \hookrightarrow (\mathcal{D}^{\bullet,\bullet}(X), \bar{\partial})$.
- Elliptic regularity theory: The above inclusion is a quasi-isomorphism, i.e. we can compute the Dolbeault cohomology of X by currents.

Holomorphic function and Poincaré-Lelong formula

- Let f be a generically nonvanishing holomorphic function on X.
- Let Z_f be the zero locus of f hence we have a $(1,1)$-current $[Z_f]$.
- $\log|f|^2$ is locally integrable.
- Hence we can define a $(1,1)$-current $\bar{\partial}\partial\log|f|^2$.

Theorem (Poincaré-Lelong formula)

We have an equality of currents

$$\frac{1}{2\pi i}\bar{\partial}\partial\log|f|^2 = [Z_f].$$

More on Poincaré-Lelong formula

- We know that $\bar{\partial}\partial = -\partial\bar{\partial}$.
- For a holomorphic function f, we have $\bar{\partial} f = 0$, $\partial \bar{f} = 0$, hence
$$\bar{\partial}\partial f = 0 \text{ and } \bar{\partial}\partial \bar{f} = 0.$$
- Conceptually we have
$$\begin{aligned}\bar{\partial}\partial \log|f|^2 &= \bar{\partial}\partial(\log f + \log \bar{f}) = \bar{\partial}\Big(\frac{\partial f}{f} + \frac{\partial \bar{f}}{\bar{f}}\Big)\\ &= \bar{\partial}\Big(\frac{1}{f}\Big) \wedge \partial f + 0 = \bar{\partial}\Big(\frac{1}{f}\Big) \wedge df.\end{aligned}$$

Problem

In general $\frac{1}{f}$ is not locally integrable, so $\frac{1}{f}$ and $\bar{\partial}(\frac{1}{f})$ are not currents on X in the naive sense.

Residue current of a function

- [Dolbeault, 1971] and [Herrera and Lieberman, 1971] solved this problem by defining the **principle value current** $\frac{1}{f}$ and the **residue current** $\bar{\partial}(\frac{1}{f})$ as
$$(\frac{1}{f})(\omega) := \lim_{\epsilon \to 0} \int_{|f| > \epsilon} \frac{\omega}{f}, \text{ and } \bar{\partial}(\frac{1}{f})(\psi) := \lim_{\epsilon \to 0} \int_{|f| = \epsilon} \frac{\psi}{f}$$
for a testing $2n$-form ω and $(2n-1)$-form ψ.
- $\bar{\partial}(\frac{1}{f})$ is a well-defined $(0,1)$-current, which we also denote by R_f.

A baby example

- Let $X = \mathbb{C}$ and $f = z$.
- If we write the testing $(1,0)$-form θ as $\theta = s(z)dz$, then a polar coordinate computation shows $\frac{1}{2\pi i}\bar{\partial}(\frac{1}{z})(\theta) = s(0)$.
- For a testing function $s(z)$ we have
$$\frac{1}{2\pi i}\bar{\partial}(\frac{1}{z}) \wedge (dz)(s(z)) = \frac{1}{2\pi i}\bar{\partial}(\frac{1}{z})(s(z) \wedge dz) = s(0).$$
- On the other hand $Z_f = \{0\}$
- We checked $\bar{\partial}(\frac{1}{z}) \wedge (dz) = \bar{\partial}\partial \log |z|^2$ and Poincaré-Lelong formula by hand.

Duality principle

Proposition (Duality principle)

A holomorphic function g on X is a multiple of f if and only if $g\bar{\partial}(\frac{1}{f}) = 0$ as a current.

- In the case $X = \mathbb{C}$ and $f = z$, the duality principle says: $g(z)$ is a multiple of z if and only if $g(0) = 0$.
- The proof of the results in [Dolbeault, 1971] and [Herrera and Lieberman, 1971] in the general case depends on *Hironaka's desingularization theorem*.

Residue current of a collection of functions

- Let $f = (f_1, \ldots, f_m)$ be a collection of holomorphic functions.
- A path $\epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_m(t))$ in $\mathbb{C}^m$ is called *admissible* if
$$\lim_{t\to 0} \epsilon_m(t) = 0, \text{ and } \lim_{t\to 0} \frac{\epsilon_j(t)}{(\epsilon_{j+1}(t))^q} = 0, \; j = 1, \ldots, m-1,$$
for any positive integer q.
- [Coleff and Herrera, 1978]: We can define the residue current R_f of f as
$$R_f(\psi) := \lim_{t\to 0} \int_{|f_1|=\epsilon_1(t),\ldots,|f_m|=\epsilon_m(t)} \frac{\psi}{f_1 \cdot \ldots \cdot f_m}$$
where $\epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_m(t))$ is an admissible path and ψ is a test $(2n-m)$-form.
- R_f is a well-defined $(0, m)$-current. Heuristically we can consider it as the (noncommutative) wedge product $R_f = \bar{\partial}(\frac{1}{f_1}) \wedge \ldots \wedge \bar{\partial}(\frac{1}{f_m})$.

Review: currents valued in vector bundles

Let E be a C^∞-vector bundle on X with dual bundle E^*.

- A (p,q)-current valued in E is a continuous linear map from $\Omega_c^{n-p,n-q}(X,E^*)$ to $\mathbb{C}$.
- A (p,q)-current valued in $\mathrm{End}(E)$ is a continuous linear map from $\Omega_c^{n-p,n-q}(X,\mathrm{End}(E))$ to $\mathbb{C}$.
- We can define wedge products, differential operators, traces, etc. on bundle-valued current as before.

Complexes of holomorphic vector bundles and minimal right inverse

[Andersson and Wulcan, 2007]

- Let

$$\xi : 0 \to E_{-N} \xrightarrow{\phi_{-N}} E_{-N+1} \xrightarrow{\phi_{-N+1}} \ldots \xrightarrow{\phi_{-1}} E_0 \to 0$$

  be a bounded complex of holomorphic vector bundles.

- We equip each E_i with a Hermitian metric.
- For each $i = -1, \ldots, -N$, let $\sigma_i : E_{i+1} \to E_i$ be the **minimal right inverse** of ϕ_i.
- Minimal right inverse is defined by the following properties:

$$\phi_i \sigma_i|_{\mathrm{im}\phi_i} = \mathrm{id}_{\mathrm{im}\phi_i},\ \sigma_i|_{(\mathrm{im}\phi_i)^\perp} = 0, \text{ and } \mathrm{im}\sigma_i \perp \ker \phi_i \Rightarrow \sigma_{i-1}\sigma_i = 0.$$

- Minimal right inverse exists.

Minimal right inverse: an example

- $\underline{\mathbb{C}}^m$ the m-dimensional trivial vector bundle on X equipped with the standard Hermitian metric.
- A map $\phi : \underline{\mathbb{C}}^m \to \underline{\mathbb{C}}$ is given by $\phi = (f_1, \ldots, f_m)$ where $f_1, \ldots, f_m$ are C^∞-functions on X.
- If all f_i's are identically 0 on X, then the maximal rank of im ϕ is 0, hence $Z = \emptyset$ and $\sigma \equiv 0$.
- If some f_i's are not identically 0 on X, then the maximal rank of im ϕ is 1, hence $Z = \{x \in X | f_1(x) = \ldots = f_m(x) = 0\}$ and

$$\sigma(x) = \begin{cases} 0 & x \in Z \\ \frac{1}{\sum_{i=1}^m |f_i|^2} \begin{pmatrix} \overline{f_1} \\ \ldots \\ \overline{f_m} \end{pmatrix} & x \in X \backslash Z. \end{cases}$$

Minimal right inverse: properties

- σ_i could be singular, i.e., it could go to ∞.
- Let Z be the union of all singular locus of the σ_i's. Z has positive codimension in X.
- We are mostly interested in the case that ξ is acyclic on $X \backslash Z$.
- $\bar{\partial}\sigma_i$ may be nonzero, even when restricted to $X \backslash Z$.
- Notation

$$E^{\bullet} := \bigoplus_{i=-N}^{-1} E_i,$$

$$\phi := \phi_{-N} + \phi_{-N+1} + \ldots + \phi_{-1},$$

$$\sigma := \sigma_{-N} + \sigma_{-N+1} + \ldots + \sigma_{-1}.$$

The preimage problem

- If ξ is acyclic, then we can check $\sigma\phi + \phi\sigma = \mathrm{id}_{E^\bullet}$.

Question

If ξ is acyclic, then for $e \in E_0$ a holomorphic section, can we find a holomorphic element $x \in E_{-1}$ such that

$$\phi x = e?$$

- Naive answer: $x = \sigma e$, hence

$$\phi x = \phi\sigma e = (\sigma\phi + \phi\sigma)e = e.$$

- Problem: x is not holomorphic.

The construction of u

- On $X \backslash Z$ we define the $\mathrm{End}(E^\bullet)$-valued form
$$u := \sigma(\mathrm{id}_{E^\bullet} - \bar\partial\sigma)^{-1} = \sigma + \sigma(\bar\partial\sigma) + \sigma(\bar\partial\sigma)^2 + \dots$$
- If ξ is acyclic, then for $e \in E_0$ a holomorphic section, the equation
$$(\phi - \bar\partial)x = e$$
has a solution in $\oplus_{i=-N}^{-1}\Omega^{0,-i-1}(X, E_i)$ given by ue.
- $([\phi, u] - \bar\partial u)e = e$
- The $\bar\partial$-operator is locally exact.
- Locally on X we can find $\tilde{x} \in E_{-1}$ such that $\tilde{x}$ is holomorphic and $\phi\tilde{x} = e$.
- u plays the role of $\frac{1}{f}$ before.

Almost semi-meromorphic and pseudomeromorphic currents

We follow [Andersson and Wulcan, 2010, Andersson and Wulcan, 2018].

- We can define **almost semi-meromorphic** currents on X, which generalize principal value currents $[\frac{1}{f}]$.
- We can define **pseudomeromorphic** currents on X, which generalize residue currents $\bar{\partial}(\frac{1}{f_1}) \wedge \ldots \wedge \bar{\partial}(\frac{1}{f_m})$.
- We can extend u to a $\mathrm{End}(E^\bullet)$-valued, almost semi-meromorphic current U on X.
- We can define a current $[\phi, U] - \bar{\partial}U$, which is a $\mathrm{End}(E^\bullet)$-valued, pseudomeromorphic current U on X.

Residue current of ξ

- The **residue current** R_ξ of the cochain complex ξ is an $\mathrm{End}(E^\bullet)$-valued, pseudomeromorphic current defined by

$$R_\xi := \mathrm{id}_{E^\bullet} - [\phi, U] + \bar{\partial} U.$$

- R_ξ measures how the cochain complex ξ fails to be acyclic.
- It is easy to see that when ξ is the complex $\underline{\mathbb{C}} \xrightarrow{f} \underline{\mathbb{C}}$, then R_ξ reduce to R_f.

Some notations

- Let $R_{\xi}^{i \to j}$ denote the component of R_{ξ} that maps E_i to E_j.
- For a coherent sheaf $\mathcal{F}$, we define its **cycle** as the current

$$[\mathcal{F}] := \sum_i m_i [Z_i]$$

  where Z_i is the irreducible component of the support of $\mathcal{F}$ and m_i is the multiplicity of Z_i in $\mathcal{F}$.
- We say $\mathcal{F}$ has pure codimension p if supp$\mathcal{F}$ has pure codimension p.

Duality principle and generalized Poincaré-Lelong

Theorem (Duality principle, [Andersson and Wulcan, 2007])

If ξ is acyclic on $X \backslash Z$, then for a holomorphic section e of E^0, $R_\xi e = 0$ if and only if e can be locally written as ϕx where x is a holomorphic section of E_{-1}.

Theorem (Generalized Poincaré-Lelong formula, [Lärkäng and Wulcan, 2021])

If ξ is a resolution of a coherent sheaf $\mathfrak{F}$ with pure codimension p. Then

$$\frac{1}{(2\pi i)^p p!} tr(D\phi_{-1}) \ldots (D\phi_{-p}) R_\xi^{0 \to -p} = [\mathfrak{F}].$$

where D is a connection on $E^\bullet$ which is compatible with $\bar{\partial}$.

Question

What if $\mathfrak{F}$ does not have a locally free resolution on X?

Coherent sheaves

Definition

Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$ is called a **coherent sheaf** if it satisfies the following two conditions.

- $\mathfrak{F}$ is of finite type over $\mathcal{O}_X$, that is, every point in X has an open neighborhood U in X such that there is a surjective morphism $\mathcal{O}_X^n|_U \twoheadrightarrow \mathfrak{F}|_U$ for some natural number n;
- for **any** open set $U \subseteq X$, **any** natural number n, and **any** morphism $\varphi : \mathcal{O}_X^n|_U \to \mathfrak{F}|_U$ of $\mathcal{O}_X$-modules, the kernel of φ is of finite type.

More on coherent sheaves

- The category of coherent sheaves is abelian.
- Oka's coherence principle: Let $(X, \mathcal{O}_X)$ be a complex manifold with holomorphic functions. Then $\mathcal{O}_X$ itself is coherent. $\Rightarrow$ Any finitely generated locally free sheaf (i.e. finite dimensional holomorphic vector bundles) is coherent.
- Cohomology sheaves of cochain complexes of finite dimensional holomorphic vector bundles must be coherent.
- Not all coherent sheaves are locally free, for example, skyscraper sheaves, ideal sheaves, etc.
- Example: Let $X = \mathbb{C}$ and consider the cochain complex

$$0 \to \mathcal{O}_{\mathbb{C}} \xrightarrow{z} \mathcal{O}_{\mathbb{C}} \to 0.$$

Its cohomology sheaves are 0 and $\mathcal{O}_{\{0\}}$.

Derived category of coherent sheaves

- $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$: the bounded derived category of coherent sheaves on X. $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is a triangulated category.
- Objects: bounded complexes of $\mathcal{O}_X$-modules with coherent cohomologies.
- Morphisms: have the universal property that quasi-isomorphisms are invertible in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Coherent sheaves and holomorphic vector bundles

- Syzygy theorem: Let $\mathfrak{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for a complex manifold X. Every point in X has an open neighborhood U in X such that there is a locally free resolution $E^\bullet \xrightarrow{\sim} \mathfrak{F}|_U$ where $E^\bullet$ is a bounded complex of holomorphic vector bundles on U.
- It does not hold if X is singular.
- The locally free resolution does not always exist globally on X if X is not projective.
- Counter example: X is a generic complex torus with dimension ≥ 3. $\mathfrak{F}$ is a point sheaf. See [Voisin, 2002].

Holomorphic vector bundles

- Let X be a complex manifold.

Theorem (Koszul-Malgrange theorem)

A holomorphic vector bundles over a complex manifold X is equivalently a complex C^∞-vector bundle E which are equipped with a $\bar{\partial}$-connection

$$\nabla^{E\prime\prime} : E \to \Omega^{0,1}(X, E).$$

such that $(\nabla^{E\prime\prime})^2 = 0$.

A holomorphic map from E to F is equivalently a C^∞-map $f : E \to F$ such that

$$\nabla^{F\prime\prime} f = f \nabla^{E\prime\prime}.$$

Complexes of holomorphic vector bundles

In the view of the Koszul-Malgrange theorem, a cochain complex of finite dimensional holomorphic vector bundles corresponds to the following data:

- a $\mathbb{Z}$-graded C^∞-vector bundle $E^\bullet$;
- a C^∞-map $v : E^i \to E^{i+1}$ and a map $\nabla^{E^i\prime\prime} : E^i \to \Omega^{0,1}(X, E^i)$ such that
$$v^2 = 0, (\nabla^{E^i\prime\prime})^2 = 0 \text{ and } [v, \nabla^{E^i\prime\prime}] = 0.$$
In other words $(v + \nabla^{E^i\prime\prime})^2 = 0$.
- We modified the $\pm$ sign for later applications.
- $A^{E^\bullet\prime\prime} := v + \nabla^{E^i\prime\prime}$ is an example of antiholomorphic flat superconnections defined later.

Antiholomorphic superconnection

Definition (Antiholomorphic superconnection)

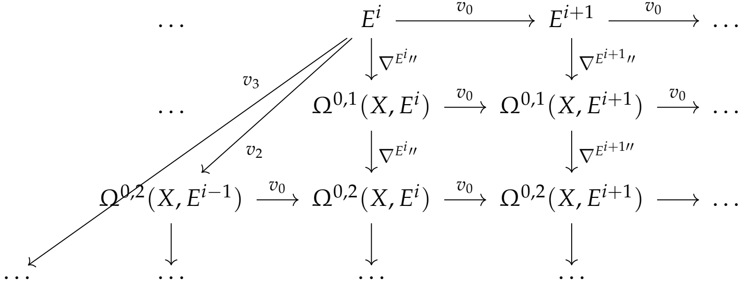
Let X be a complex manifold and $E^\bullet$ be a $\mathbb{Z}$-graded bounded C^∞-vector bundles on X. An **antiholomorphic superconnection** $A^{E^\bullet\prime\prime}$ is a differential operator of total degree 1 acting on $\Omega^{0,\bullet}(X, E^\bullet)$ which satisfies the Leibniz rule

$$A^{E^\bullet\prime\prime}(\omega e) = \bar{\partial}(\omega)\cdot e + (-1)^{|\omega|}\omega \wedge A^{E^\bullet\prime\prime}(e),\ \omega \in \Omega^{0,\bullet}(X),\ e \in E^\bullet.$$

- By the Leibniz rule $A^{E^\bullet\prime\prime}$ is determined by its restriction to $E^\bullet$.
- $A^{E^\bullet\prime\prime} = v_0 + \nabla^{E^\bullet\prime\prime} + v_2 + \ldots$ where $v_i \in \Omega^{0,i}(X, \mathrm{End}^{1-i}(E^\bullet))$ and $\nabla^{E^\bullet\prime\prime}$ is an antiholomorphic connection on each E^i.
- $v_0 : E^i \to E^{i+1}$, $v_2 : E^i \to \Omega^{0,2}(X, E^{i-1})$, ...

Antiholomorphic superconnection illustration

This is an illustration:



Cohesive modules

Definition ([Block, 2010])

A cohesive module on a complex manifold X is a $\mathbb{Z}$-graded bounded C^∞-vector bundles $E^\bullet$ together with a **flat** antiholomorphic superconnection $A^{E^\bullet\prime\prime}$. Flat means $(A^{E^\bullet\prime\prime})^2 = 0$.

- Componentwisely, flatness means that
  0. $v_0^2 = 0$: $(E^\bullet, v_0)$ is a cochain complex of C^∞-vector bundles;
  1. $[\nabla^{E^\bullet\prime\prime}, v_0] = 0$: v_0 and $\nabla^{E^\bullet\prime\prime}$ are compatible;
  2. $(\nabla^{E^\bullet\prime\prime})^2 + [v_0, v_2] = 0$: $\nabla^{E^\bullet\prime\prime}$ is flat *up to cochain homotopy*.

  ...
- In [Bismut et al., 2023] is called an *antiholomorphic flat superconnection*.

Cohesive modules as a DG-category

- Cohesive modules on X form a differential graded (DG)-category $\mathsf{B}(X)$ which has a pre-triangulated structure.
- Let $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ and $\mathcal{F} = (F^\bullet, A^{F^\bullet\prime\prime})$ be two cohesive modules on X. A morphism $\phi : \mathcal{E} \to \mathcal{F}$ of degree k is given by

$$\phi = \phi_0 + \phi_1 + \dots$$

  where $\phi_i \in \Omega^{0,i}(X, \mathrm{Hom}^{k-i}(E^\bullet, F^\bullet))$ is $C^\infty(X)$-linear.
- We can define the differential of ϕ as $d(\phi) := A^{F^\bullet\prime\prime} \circ \phi - (-1)^k \phi \circ A^{E^\bullet\prime\prime}$.
- A degree 0 closed morphism ϕ is called a *gauge equivalence* if it admits an inverse in $\mathsf{B}(X)$

Cohesive modules and cochain complexes of sheaves

For a cohesive module $\mathcal{E} = (E^\bullet, A^{E^\bullet \prime\prime})$ on X. We define its associated cochain complex of sheaves of $\mathcal{O}_X$-modules as follows:

- The sheaf $\mathfrak{E}^n$ is such that
$$\mathfrak{E}^n(U) := \bigoplus_{p+q=n} \Omega^{0,p}(U, E^q).$$
- The differential $\mathfrak{E}^n \to \mathfrak{E}^{n+1}$ is induced by $A^{E^\bullet \prime\prime}$.
  - $A^{E^\bullet \prime\prime}$ is of total degree 1;
  - $A^{E^\bullet \prime\prime}$ commutes with holomorphic functions, hence it is a map of $\mathcal{O}_X$-modules.
- Gauge equivalent cohesive modules give isomorphic cochain complexes

Gauge equivalence in local charts

We will need the following result:

Proposition (Gauge trivialization property, [Block, 2010])

For a cohesive module $\mathcal{E} = (E^\bullet, A^{E^\bullet \prime\prime})$ on X. For any $x \in X$, there exists an open neighborhood V of x and a flat $\bar{\partial}$-connection $\overline{\nabla}^{E^\bullet|_V \prime\prime}$ on $E^\bullet|_V$ such that

1. $\overline{\nabla}^{E^\bullet|_V \prime\prime}(v_0) = 0$, *i.e.* $(E^\bullet|_V, v_0 + \overline{\nabla}^{E^\bullet|_V \prime\prime})$ *is a cohesive module on V with $v_i = 0$ for all $i \geq 2$;*
2. *There exists a gauge equivalence* $J : (E^\bullet, A^{E^\bullet \prime\prime})|_V \xrightarrow{\sim} (E^\bullet|_V, v_0 + \overline{\nabla}^{E^\bullet|_V \prime\prime})$.

- Corollary: the associated cochain complex of any cohesive module has coherent cohomology sheaves.
- The map $v_0 : E^\bullet \to E^{\bullet+1}$ remains unchanged.

Cohesive modules and coherent sheaves

Theorem ([Block, 2010])

For a compact complex manifold X, $\underline{B}(X)$, the homotopy category of B(X), is triangulated equivalent to $D^b_{coh}(X)$. In other words, B(X) gives a dg-enhancement of $D^b_{coh}(X)$.

- In particular, a coherent sheaf corresponds to a cohesive module which is unique up to *homotopy equivalence*.
- Chuang, Holstein, and Lazarev ([Chuang et al., 2021]) generalized this result to non-compact complex manifolds with slightly modified $D^b_{coh}(X)$.

Applications

- We can use analytic tool to study coherent sheaves on complex manifold.
- [Qiang, 2017]: Chern-Weil theory for coherent sheaves.
- [Bismut et al., 2023] Grothendieck-Riemann-Roch theorem for coherent sheaves on complex manifolds. The proof is by the heat-kernel method.
- [Wei, 2024] Residue currents and generalized Poincaré-Lelong formula for coherent sheaves.

Some motivations on residue currents of cohesive modules

Let $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ be a cohesive module on X with $A^{E^\bullet\prime\prime} = v_0 + \nabla^{E^\bullet\prime\prime} + v_2 + \dots$.

- $(E^\bullet, v_0)$ is a cochain complex of C^∞-vector bundles on X.
- Let $(\mathfrak{E}^\bullet, A^{E^\bullet\prime\prime})$ be the associated cochain complex of sheaves of $\mathcal{O}_X$-modules as before.
- By the gauge trivialization property, whether or not $(\mathfrak{E}^\bullet, A^{E^\bullet\prime\prime})$ is acyclic only depends on $(E^\bullet, v_0)$.
- On the other hand, recall that the residue current R_ξ measures how the cochain complex ξ fails to be acyclic.
- This suggests that we can define the residue current of the cohesive module $\mathcal{E}$ via v_0.

Some more notations

Let $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ be a cohesive module on X.

- $A^{E^\bullet\prime\prime} = v_0 + \nabla^{E^\bullet\prime\prime} + v_2 + \ldots$ as before.
- Let $A^{E^\bullet\prime\prime}_{\geq 1}$ denote $\nabla^{E^\bullet\prime\prime} + v_2 + v_3 + \ldots$.
- We equip each E^i with a Hermitian metric.
- We find the minimal right inverse σ of v_0.
- Let Z be the singular locus of σ.

Proposition ([Wei, 2024])

Z is an analytic subvariety of X.

- This is a consequence of the gauge trivialization property of cohesive modules.

The construction of $u_{\mathcal{E}}$

Definition ([Wei, 2024])

Define an $\mathrm{End}(E^\bullet)$-valued form u on $X\backslash Z$ as

$$u_{\mathcal{E}} := \sigma(\mathrm{id}_{E^\bullet} + [A^{E^\bullet\prime\prime}_{\geq 1}, \sigma])^{-1} = \sigma - \sigma[A^{E^\bullet\prime\prime}_{\geq 1}, \sigma] + \sigma[A^{E^\bullet\prime\prime}_{\geq 1}, \sigma]^2 - \dots$$

- The above sum is finite by degree reason.
- We can check that if the complex of C^∞-vector bundles $(E^\bullet, v_0)$ is acyclic on $X\backslash Z$, then

$$[A^{E^\bullet\prime\prime}, u_{\mathcal{E}}] = \mathrm{id}_{E^\bullet} \text{ on } X\backslash Z.$$

  In this case $u_{\mathcal{E}}$ is a homotopy operator of $A^{E^\bullet\prime\prime}$.
- By the gauge trivialization property, the local expression of $u_{\mathcal{E}}$ is similar to the u in [Andersson and Wulcan, 2007].

The current $U_{\mathcal{E}}$

- We can extend $u_{\mathcal{E}}$ to a current $U_{\mathcal{E}}$ on X.
- $U^{\mathcal{E}}$ is an almost semimeromorphic current.
- $U^{\mathcal{E}}$ is an $\mathrm{End}(E^{\bullet})$-valued, $(0,\bullet)$-current on X with total degree -1.
- In general

$$[A^{E^{\bullet}\prime\prime}, U_{\mathcal{E}}] \neq \mathrm{id}_{E^{\bullet}} \text{ on } X$$

  even when $[A^{E^{\bullet}\prime\prime}, u_{\mathcal{E}}] = \mathrm{id}_{E^{\bullet}}$ on $X \backslash Z$.

The residue current of $\mathcal{E}$

Definition ([Wei, 2024])

We define the **residue current** $R^{\mathcal{E}}$ of a cohesive module $\mathcal{E}$ as

$$R^{\mathcal{E}} := \mathrm{id}_{E^\bullet} - [A^{E^\bullet\prime\prime}, U_{\mathcal{E}}].$$

- $R^{\mathcal{E}}$ is an $\mathrm{End}(E^\bullet)$-valued, $(0,\bullet)$-current on X with total degree 0.
- $R^{\mathcal{E}}$ is a pseudomeromorphic current.
- It is easy to check that

$$[A^{E^\bullet\prime\prime}, R^{\mathcal{E}}] = 0.$$

The dimension principle of pseudomeromorphic currents

- The *support* of a current T is defined to be the complement of the biggest open set $U \subset X$ such that $T(\omega) = 0$ whenever $\omega \in \Omega_c(U)$.

We have the following important property of pseudomeromorphic currents:

Proposition (Dimension principle, [Andersson and Wulcan, 2010])

Let T be a pseudomeromorphic $(*, q)$-current on X with support on a subvariety Z. If codim $Z \geq q + 1$, then $T = 0$.

The vanishing property of $R^{\mathcal{E}}$

- Let $R^{\mathcal{E}}_{p\to q}$ be the component of $R^{\mathcal{E}}$ that maps E^p to E^q.

Proposition (Vanishing property of $R^{\mathcal{E}}$, [Wei, 2024])

If $(E^\bullet, v_0)$ is acyclic at degrees $< n_0$, then for any $p < n_0$ and any q we have

$$R^{\mathcal{E}}_{p\to q} = 0.$$

- The proof depends on the dimension principle together with a careful investigation of the support of $R^{\mathcal{E}}_{p\to q}$.
- The vanishing property shows that $R^{\mathcal{E}}$ precisely measures how $(E^\bullet, v_0)$ fails to be acyclic.

The duality principle for cohesive modules

- Let $(E^\bullet, v_0)$ is acyclic at degrees $< n_0$.
- Let e be a section of E^{n_0} such that $A^{E^\bullet \prime\prime}(e) = 0$.

Theorem (Duality principle, [Wei, 2024])

$R^{\mathcal{E}} e = 0 \Leftrightarrow$ *there exists an* $x \in \oplus_{p+q=n_0-1}\Omega^{0,p}(X, E^q)$ *such that* $e = A^{E^\bullet \prime\prime} x$.

- The "if" part follows from the fact that $[A^{E^\bullet \prime\prime}, R^{\mathcal{E}}] = 0$.
- The "only if" part follows from the vanishing property of $R^{\mathcal{E}}$.
- Can we make x a holomorphic section? Globally it does not even make sense. But locally yes, with the help of *gauge transformations* of $\mathcal{E}$.

The current M^{ϕ} related to a morphism

- Let $\mathcal{E} = (E^{\bullet}, A^{E^{\bullet}\prime\prime})$ and $\mathcal{F} = (F^{\bullet}, A^{F^{\bullet}\prime\prime})$ be two cohesive modules on X.
- Let $\phi : \mathcal{E} \to \mathcal{F}$ be a closed degree 0 morphism.

Proposition ([Wei, 2024])

We can construct a current M^{ϕ} such that

$$R^{\mathcal{F}}\phi - \phi R^{\mathcal{E}} = A^{F^{\bullet}\prime\prime}M^{\phi} + M^{\phi}A^{E^{\bullet}\prime\prime}.$$

- M^{ϕ} is an $\mathrm{Hom}^{\bullet}(E^{\bullet}, F^{\bullet})$-valued, $(0, \bullet)$-current on X with total degree -1.
- M^{ϕ} is a pseudomeromorphic current.

The comparison formula

- Let $\phi : \mathcal{E} \to \mathcal{F}$ and $\psi : \mathcal{F} \to \mathcal{E}$ be two closed degree 0 morphisms which are homotopic inverse to each other, i.e. there exists degree -1 morphisms $\tau : \mathcal{E} \to \mathcal{E}$ and $\gamma : \mathcal{F} \to \mathcal{F}$ such that
$$\psi\phi - \mathrm{id}_{E^\bullet} = A^{E^\bullet\prime\prime}(\tau), \text{ and } \phi\psi - \mathrm{id}_{F^\bullet} = A^{F^\bullet\prime\prime}(\gamma).$$
- Let M^ϕ and M^ψ be as before.

Corollary (Comparison formula, [Wei, 2024])

The residue currents $R^{\mathcal{E}}$ and $R^{\mathcal{F}}$ are chain homotopic. Actually we have
$$R^{\mathcal{E}} - \psi R^{\mathcal{F}}\phi = A^{E^\bullet\prime\prime}(M^\psi\phi - R^{\mathcal{E}}\tau),$$
$$R^{\mathcal{F}} - \phi R^{\mathcal{E}}\psi = A^{F^\bullet\prime\prime}(M^\phi\psi - R^{\mathcal{F}}\gamma).$$

- Same result holds for gauge equivalence with $\psi = \phi^{-1}$, $\tau = 0$, $\gamma = 0$.

The generalized Poincaré-Lelong formula

- Let $\mathfrak{F}$ be a coherent sheaf on X.
- Let $\mathcal{E} = (E^\bullet, A^{E^\bullet\prime\prime})$ be a cohesive module corresponding to $\mathfrak{F}$. Recall that $A^{E^\bullet\prime\prime} = v_0 + \nabla^{E^\bullet\prime\prime} + v_2 + \dots$
- Let $\nabla^{E^\bullet\prime}$ be an arbitrary ∂-connection on $E^\bullet$ and $\nabla^{E^\bullet} := \nabla^{E^\bullet\prime} + \nabla^{E^\bullet\prime\prime}$.

Theorem (Generalized Poincaré-Lelong formula, [Wei, 2024])

Suppose the cycle $[\mathfrak{F}]$ has pure codimension $p \geq 1$. Then we have

$$\frac{1}{(2\pi i)^p p!} Tr_s((\nabla^{E^\bullet}(v_0))^p R^{\mathcal{E}}) = [\mathfrak{F}],$$

where Tr_s is the supertrace on $\mathrm{End}(E^\bullet)$.

- Proof: gauge trivialization + comparison formula reduce it to [Lärkäng and Wulcan, 2021].
- We also have a variation of non pure codimension case.

Related works and further works

- Related works:
  - [Johansson, 2023] and [Johansson and Lärkäng, 2024]: residue currents of twisted cochains.
  - [Han, 2024]: characteristic currents of cohesive modules.
- Further works:
  - Singular connections?
  - Further study of Baum-Bott residues of singular holomorphic foliations parallel to [Kaufmann et al., 2024]?

Thank you!

Andersson, M. and Wulcan, E. (2007).
Residue currents with prescribed annihilator ideals.
Ann. Sci. École Norm. Sup. (4), 40(6):985–1007.

Andersson, M. and Wulcan, E. (2010).
Decomposition of residue currents.
J. Reine Angew. Math., 638:103–118.

Andersson, M. and Wulcan, E. (2018).
Direct images of semi-meromorphic currents.
Ann. Inst. Fourier (Grenoble), 68(2):875–900.

Bismut, J.-M., Shen, S., and Wei, Z. (2023).
Coherent Sheaves, Superconnections, and Riemann-Roch-Grothendieck, volume 347 of *Progress in Mathematics*.
Birkhäuser/Springer, Cham.

Block, J. (2010).

Duality and equivalence of module categories in noncommutative geometry.

In *A celebration of the mathematical legacy of Raoul Bott*, volume 50 of *CRM Proc. Lecture Notes*, pages 311–339. Amer. Math. Soc., Providence, RI.

Chuang, J., Holstein, J., and Lazarev, A. (2021).

Maurer-Cartan moduli and theorems of Riemann-Hilbert type.

Appl. Categ. Structures, 29(4):685–728.

Coleff, N. R. and Herrera, M. E. (1978).

Les courants résiduels associés à une forme méromorphe, volume 633 of *Lecture Notes in Mathematics*.

Springer, Berlin.

Dolbeault, P. (1971).

Courants résidus des formes semi-méromorphes.

In *Séminaire Pierre Lelong (Analyse) (année 1970)*, volume Vol. 205 of *Lecture Notes in Math.*, pages 56–70. Springer, Berlin-New York.

Han, Z. T. (2024).
Characteristic currents on cohesive modules.
arXiv preprint arXiv:2404.09439.

Herrera, M. and Lieberman, D. (1971).
Residues and principal values on complex spaces.
Math. Ann., 194:259–294.

Johansson, J. (2023).
A residue current associated with a twisting cochain: duality and comparison formula.
arXiv preprint arXiv:2306.02458.

Johansson, J. and Lärkäng, R. (2024).
An explicit isomorphism of different representations of the Ext functor using residue currents.
Ark. Mat., 62(2):413–443.

Kaufmann, L., Lärkäng, R., and Wulcan, E. (2024).
Global chern currents of coherent sheaves and Baum Bott currents.
arXiv preprint arXiv:2404.14585.

Lärkäng, R. and Wulcan, E. (2021).
Residue currents and cycles of complexes of vector bundles.
Ann. Fac. Sci. Toulouse Math. (6), 30(5):961–984.

Qiang, H. (2017).
Bott-Chern Characteristic Forms and Index Theorems for Coherent Sheaves on Complex Manifolds.
ProQuest LLC, Ann Arbor, MI.
Thesis (Ph.D.)–University of Pennsylvania.

Voisin, C. (2002).
A counterexample to the Hodge conjecture extended to Kähler varieties.
Int. Math. Res. Not., (20):1057–1075.

Wei, Z. (2024).
Residue currents of cohesive modules and the generalized Poincaré-Lelong formula on complex manifolds.
arXiv preprint arXiv:2409.17362.