SEMINAR ON QUILLEN K-THEORY

1. Homotopy Theory

1.1. **Compactly Generated Spaces.** We consider *compactly generated spaces.* The problem is that we want *adjoints* in our category. That is, given a map $f : X \times Y \to Z$, we would like to define

$$\widetilde{f} : X \to C(Y, Z),$$

the latter with the compact-open topology. In general $\widetilde{f}$ will not be continuous if f is. For example, if G acts on Z, we would like a map $G \to \mathrm{Homeo}(Z, Z)$. Similarly, given a homotopy $f : X \times I \to Y$, the maps

$$\begin{aligned} \widetilde{f} &: X \to Y^I = C(I, Y) \\ \widehat{f} &: I \to Y^X = C(X, Y) \end{aligned}$$

are interesting. Also, evaluation maps

$$C(Y, Z) \times Y \to Z$$

should be continuous.

The category $\mathcal{K}$ is the category of compactly generated spaces (a subcategory of topological spaces). We say X is *compactly generated* if

(1) X is Hausdorff
(2) $A \subset X$ is closed iff $A \cap C$ is closed for every compact C in X.

For example, locally compact Hausdorff spaces (manifolds), metric spaces, and CW complexes with finitely many cells in each dimension are all in $\mathcal{K}$.

Given a Hausdorff space Y, you can modify its topology (to $\mathcal{K}(Y)$) so that it is in $\mathcal{K}$. That is, say that A is closed iff $A \cap C$ is closed for all compact C in the original topology. You can also show that $C(X, \mathcal{K}(Y)) \cong C(X, Y)$.

Note that if $X, Y \in \mathcal{K}$, it is not necessarily true that $X \times Y$ is in $\mathcal{K}$. We will define $\mathcal{K}(X \times Y)$ to be the product in $\mathcal{K}$. However, if X is locally compact and Y is compactly generated, then $X \times Y$ is compactly generated. Note that if X, Y are compactly generated, $C(X, Y)$ is not necessarily c.g. (in compact open topology). However, let $\mathrm{Map}(X, Y) = \mathcal{K}(C(X, Y))$. All adjoints are continuous using Map. That is, if $f : X \times Y \to Z$ is continuous, then

$$\widehat{f} : X \to \mathrm{Map}(Y, Z)$$

is continuous.

1.2. **Fibrations.** We say that $p : E \to B$ is a fibration if it satisfies the homotopy lifting property. That is, a homotopy $Z \times I \to B$ together with a map $Z \times \{0\} \to B$ can be lifted to $Z \times I \to E$ such that the corresponding diagram commutes:

$$\begin{array}{ccc} Z \times \{0\} & \to & E \\ \downarrow & \nearrow & \downarrow \\ Z \times I & \to & B \end{array}$$

For example, covering spaces and fiber bundles are fibrations, because fibrations are local. For example, the flat bow tie is a fibration but not a fiber bundle.

Given a fibration $p : E \to B$, it is easy to show the pullback is also a fibration.

$$\begin{array}{ccc} p^*E & \to & E \\ \downarrow & & \downarrow \\ X & \to & B \end{array}$$

Given a fibration, the fiber is $F_b = p^{-1}(b)$.

Theorem 1.1. *If B is path connected, all fibers are homotopy equivalent. In fact, any path in B from a to b determines a unique homotopy class of maps from F_b to F_a. In particular, $\pi_1(B, b) \to \{$self homotopy equivalences of $F_b\}$ is an isomorphism of groups.*

Proof. Given a fibration, and a path $\alpha : I \to B$ from a to b in B. Consider

$$\begin{array}{ccc} F_b \times \{0\} & \to & E \\ \downarrow & \nearrow & \downarrow \\ F_b \times I & \to & B \end{array}$$

where F_b maps to b on the bottom row. Then $\alpha : F_b \to F_a$ is defined to be the image of $(f, 1)$ in E via the diagonal map. Any two such maps have to be homotopic, by a similar argument. Given two such paths α_1 and α_2, we get a diagram

$$\begin{array}{ccccc} \begin{array}{c} F_b \times I \times \{0\} \\ \cup F_b \times \{0,1\} \times I \end{array} & = & F_b \times I \times \{0\} & \to & E \\ & & \downarrow & \nearrow & \downarrow \\ & & F_b \times I \times I & \to & B \end{array}$$

and the lift provides the homotopy.

Next, given a path γ from a to b, we have a homotopy $\gamma_* : F_b \to F_a$, and also $\gamma_*^{-1} : F_a \to F_b$. Since $\gamma \circ \gamma^{-1}$ is the constant path over b, $(\gamma \circ \gamma^{-1})_* = \gamma_* \circ \gamma_*^{-1} = \mathbf{1}$, so the fibers are homotopy equivalent. □

1.3. **Path space fibrations.** Let (Y, y_0) be a based, path-connected space. Let the path space $P_{y_0}Y$ be the space of paths $\gamma : [0,1] \to Y$ such that $\gamma(0) = y_0$. So

$$P_{y_0}Y = \mathrm{Map}(I, 0; Y, y_0) \subset \mathrm{Map}(I; Y).$$

Also, let $\Omega_{y_0}Y \subset P_{y_0}Y$ be the corresponding loop space; that is,

$$\Omega_{y_0}Y = \mathrm{Map}(I, 0, 1; Y, y_0, y_0).$$

Let Y^I denote the free path space

$$Y^I = \mathrm{Map}(I; Y).$$

Observe that the choice of a base point does not matter:

$$\begin{aligned} P_{y_0}Y &\sim {}_{h.e.}P_{y_1}Y; \\ \Omega_{y_0}Y &\sim {}_{h.e.}\Omega_{y_1}Y. \end{aligned}$$

Theorem 1.2. (1) *$p : Y^I \to Y$ defined by $p(\gamma) = \gamma(1)$ is a fibration.*
(2) *$p : Y^I \to Y$ is a homotopy equivalence.*
(3) *$p : P_{y_0}Y \to Y$ is a fibration (with fiber $\Omega_{y_0}Y$).*
(4) *$P_{y_0}Y$ is contractible.*

Proof. See any homotopy theory book. Contractible-ness: reeling the paths in. □

1.4. **Fiber homotopy equivalences.** Given fibrations $E, E' \to B$, then a fiber homotopy equivalence is a map

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{id} & B \end{array}$$

where f is a homotopy equivalence that respects fibers (at all stages).

1.5. **Replacing continuous functions by fibrations.** The goal will be to replace any arbitrary continuous function $f : X \to Y$ by a fibration. That is, a diagram of the form

$$\begin{array}{ccc} & & E \\ & \nearrow_{h.e.} & \downarrow^{p} \\ X & \xrightarrow{f} & Y \end{array}$$

Theorem 1.3. *This can be done.*

Proof. Consider

$$\begin{array}{ccc} P_f := f^*Y^I & \to & Y^I \\ \downarrow & & \downarrow^{q} \\ X & \xrightarrow{f} & Y \end{array}$$

Note that $P_f = \left\{(x, \gamma) : x \in X, \gamma \in Y^I, f(x) = \gamma(0)\right\}$. Consider $p : P_f \to Y$, where $p((x, \gamma)) = \gamma(1)$. Claims:

(1) $p : P_f \to Y$ is a fibration
(2) $X \sim_{h.e.} P_f$. (proof: project $P_f \to X$.)

Note that (1) comes from

$$\begin{array}{ccc} * \times 0 & \to & P_f \\ \downarrow & & \downarrow^{p} \\ * \times I & \xrightarrow{H} & Y \end{array}$$

□

What if $f : X \to Y$ was originally a fibration? Then the new fibration is fiber homotopy equivalent to the original.

1.6. **Cofibrations.** A map $A \to X$ is a *cofibration* if it satisfies the following homotopy extension property. If we have the diagram

$$\begin{array}{ccc} A \times 0 & \hookrightarrow & A \times I \\ \downarrow & & \downarrow \\ X \times 0 & \hookrightarrow & X \times I \end{array}$$

and maps from $A \times I$ and $X \times 0$ to a space Z, then we can extend the homotopy to a map from $X \times I$ to Z.

Why are cofibrations dual to fibrations? Fibration picture: $f : X \to Y$

$$\begin{array}{ccc} X & \xleftarrow{\text{eval at } 0} & X^I \\ \uparrow & \nearrow & \downarrow^{f} \\ Z & \to & Y^I \end{array}$$

Cofibration picture: $f : Y \to X$

$$\begin{array}{ccc} X & \to & X \times I \\ \downarrow & \swarrow & \uparrow f \times \mathbf{1} \\ Z & \leftarrow & Y \times I \end{array}$$

Remark 1.4. *Any $f : X \to Y$ is up to homotopy an inclusion, by forming the mapping cylinder. That is, the mapping cylinder*

$$M_f := (X \times I) \amalg Y \diagup (x, 1) \tilde{} f(x)$$

is homotopy equivalent to Y. Proof: step on the hat. It is a deformation retract.

In fact, the inclusion $X \subset M_f$ is a cofibration. The proof of this is the following more general fact.

Definition 1.5. (1) *(X, A) is a DR pair ((strong) deformation retract) if X strongly d.r.'s to A.*
(2) *(X, A) is an NDR pair (neighborhood deformation retract) if there is a homotopy $H : X \times I \to X$ such that there exists a neighborhood $U \supset A$ and $H|_{U \times I}$ d.r.'s U to A.*

Theorem 1.6. *The following are equivalent:*
(1) *(X, A) is an NDR pair*
(2) *$(X \times I, (X \times 0) \cup (A \times I))$ is a DR pair*
(3) *$(X \times 0) \cup (A \times I)$ is a retract of $X \times I$*
(4) *$i : A \hookrightarrow X$ is a cofibration.*

Example 1.7. *If X is a CW complex and A is a subcomplex, then (X, A) is an NDR pair.*

1.7. **Homotopy classes of mappings.**

Definition 1.8. *$[X, Y]$ is the homotopy classes of maps $X \to Y$.*
$[X, Y]_0$ is the homotopy classes of basepoint-preserving maps $X \to Y$.

Definition 1.9. *$A \xrightarrow{f} B \xrightarrow{g} C$ (spaces and continuous maps) is* exact *if $g^{-1}(c_0) = f(A)$.*

Theorem 1.10. (1) *If $p : E \to B$ is a fibration with fiber F and B is path connected, then the following sequence is exact*

$$[Y, F] \to [Y, E] \to [Y, B]$$

for any compactly generated Y.
(2) *If $A \hookrightarrow X$ is a cofibration, then*

$$[X \diagup A, Y] \to [X, Y] \to [A, Y]$$

is exact.

Proof. (1) Pretty simple (homotopy lifting).
(2) Cofibration property. □

Theorem 1.11. $\mathrm{Map}(X \wedge Y, Z)_0 \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)_0)_0$
$\mathrm{Map}(SX, Y)_0 \cong \mathrm{Map}(X, \Omega Y)_0$.

1.8. **Fibration and cofibration sequences.** Start with a fibration $F \to E \xrightarrow{f} B$. Next, make the fiber inclusion a fibration (replacing F with P_f) to get

$$... \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \xrightarrow{f} B.$$

This is exact.

Next, look at

$$... \to \left[S^0, \Omega^2 B\right] \to \left[S^0, \Omega F\right] \to \left[S^0, \Omega E\right] \to \left[S^0, \Omega B\right] \to \left[S^0, F\right] \to \left[S^0, E\right] \to \left[S^0, B\right]$$

Then

$$... \to \pi_2 (B) \to \pi_1 (F) \to \pi_1 (E) \to \pi_1 (B) \to \pi_0 (F) \to \pi_0 (E) \to \pi_0 (B)$$

is exact. Note that the map from

$$\pi_n (B) \to \pi_{n-1} (F)$$

is $S^{n-1} \times I \to$collapse to $S^n \xrightarrow{\phi} B$ (preserving base points). Now, since you have a fibration, you can lift the zero end to a point in the fiber. You can lift to a homotopy upstairs, so that it lands in $\pi_{n-1} (E)$, but since it pushes down trivially it is in $\pi_{n-1} (F)$.

The cofibration sequence: if (X, A) is a cofibration,

$$A \hookrightarrow X \to X / A \to \Sigma A \to \Sigma X \to \Sigma (X / A) \to ...$$

is exact. Then we also have

$$[A, Y] \leftarrow [X, Y] \leftarrow [X / A, Y] \leftarrow [\Sigma X, Y] \leftarrow ...$$

is exact. Taking Y be a $K (\pi, n)$, then this will give the exact cohomology sequence.

Note that $\pi_n (X, A) = [D^n, S^{n-1}; X, A]$. Then from this you get the long exact homotopy sequence:

$$\pi_n (A) \to \pi_n (X) \to \pi_n (X, A) \to \pi_{n-1} (A) \to ...$$

1.9. **Homotopy groups.** Note that $\pi_n (X, x_0) = [S^n, X]_0 = [I^n, \partial I^n; X, x_0]$. The addition structure is defined via $I^n \cup I^n \cong \left[0, \frac{1}{2}\right] \times I^{n-1} \cup \left[\frac{1}{2}, 1\right] \times I^{n-1} = I^n$. For $n = 0$, this is just a set. For $n \geq 1$ it is a group. For $n \geq 2$ it is an abelian group (more room to move domains around). Note that $\pi_n (X, x_0)$ is a module over $\pi_1 (X, x_0)$ (squish a little tail out of the sphere and follow the loop with this tail).

Relative homotopy groups: $\pi_n (X, A, x_0) = [I^n, \partial I^n, e_0; X, A, x_0]$.

You don't generally get excision (if it were, it would be a homology theory), but stable homotopy theory is a homology theory. Stable homotopy:

$$\pi_n^s (X) = \lim_{k \to \infty} \pi_{n+k} \left(S^k X\right).$$

The coefficient group is the stable n-stem

$$\pi_n^s (\mathrm{pt}+) = \lim_{k \to \infty} \pi_{n+k} \left(S^k S^0\right).$$

Theorem 1.12. *There is a long exact sequence*

$$\to \pi_n (A) \to \pi_n (X) \to \pi_n (X, A) \xrightarrow{\partial} \pi_{n-1} (A) \to ...$$

where ∂ is the boundary map (in more than one way).

One may prove this from the fiber sequence.

Definition 1.13. *X is called n-connected if $\pi_i (X) = 0$ for $i \leq n$.*

Note that $\pi_n(X) \cong \pi_{n+1}(SX)$ if X is "sufficiently highly" m-connected. This causes the excision to work.

Example 1.14. *Let's compute $\pi_3(S^2)$. Note that $\pi_n(S^n) \cong \mathbb{Z}$ (Hopf index). Given the Hopf fibration $S^3 \to S^2$ with S^1 fibers, we have*

$$0 = \pi_3\left(S^1\right) \to \pi_3\left(S^3\right) \to \pi_3\left(S^2\right) \to \pi_2\left(S^1\right) = 0,$$

where we have used the fact that if $\widetilde{X}$ is a covering of X, then $\pi_n\left(\widetilde{X}\right) = \pi_n(X)$ is $n > 1$. Thus,

$$\pi_3\left(S^2\right) = \mathbb{Z}.$$

1.10. **Hurewicz map.** The Hurewicz map $\rho : \pi_n(X) \to H_n(X)$ is defined as follows. If $[f] \in \pi_n(X)$,

$$\rho([f]) = f_*([S^n]).$$

Here $[S^n]$ is the fundamental class in $H_n(S^n)$. There is also relative version

$$\pi_n(X, A, x_0) \to H_n(X, A).$$

These Hurewicz maps commute with the corresponding maps from the long exact homology and homotopy sequences.

Theorem 1.15. *(Hurewicz Theorem) Let $n > 0$, suppose that X is path connected. If $\pi_k(X, x_0) = 0$ if $k < n$, then $H_k(X)$ for $0 < k < n$, and $\rho_* : \pi_n(X, x_0) \to H_n(X)$ is an isomorphism if $n > 1$. (If $n = 1$, you get merely a surjection, and $H_1 = \frac{\pi_1}{[\pi_1, \pi_1]}$.)*

Note: one uses this in surgery theory. You kill off homotopy groups by cutting out spheres and replace it with the reverse....

The relative version of the Hurewicz Theorem is a little more complicated. With similar assumptions, one obtains $H_n(X, A) = \pi_n^+(X, A) = \pi_n(X, A) \diagup$action of $\pi_1(A)$.

Definition 1.16. *We say that $f : X \to Y$ is n-connected if $\pi_k(Y, X) := \pi_k(M_f, X) = 0$ if $k \leq n$.*

Theorem 1.17. *(Whitehead Theorem)*

1. *If $f : X \to Y$ is n-connected, then $f_* : H_q(X) \to H_q(Y)$ is an isomorphism if $q < n$ and an epimorphism if $q = n$.*
2. *If X and Y are simply connected and f_* is an isomorphism for all $q < n$ and an epimorphism for $q = n$, then f is n-connected.*
3. *If X and Y are imply connected and $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n and X and Y are CW complexes, then f is a homotopy equivalence.*

Caveat: need f. The counterexample: $X = S^4 \vee (S^2 \times S^2)$ and $Y = \mathbb{C}P^2 \vee \mathbb{C}P^2$ are not homotopy equivalent.

2. Classifying Spaces

2.1. **G-spaces and G-bundles.** Let G be a topological group. Let X be a topological space that is a right G-space.

$$X \times G \to X$$

We say X is a free G-space if $xg = x$ for some $x \in X$ implies $g = e$.

Let

$$X^* = \{(x, xg) : x \in X, g \in G\}.$$

Define

$$\tau : X^* \to G$$

so that

$$x\tau(x, x') = x'.$$

This is called the *translation function* and is well-defined for free actions.

If X is a free G-space and τ is continuous, we say that X is a principal G-space. For example:

Example 2.1. *If B is any topological space, G is any group, $B \times G$ is a principal G-space with $(b, g)\, g' = (b, gg')$, so $\tau((b, g), (b, g')) = g^{-1}g'$.*

Example 2.2. *If Γ is a topological group, and $G \le \Gamma$ is a closed subgroup. Let $X = \Gamma$ is a right G-space. $\tau(\gamma, \gamma') = \gamma^{-1}\gamma'$.*

Given G-spaces X and Y, $h : X \to Y$ is a G-map if h is continuous and

$$h(xg) = h(x)\, g$$

for all $x \in X$, $g \in G$.

If X is a G-space, we say $x\tilde{\ }x'$ is $xg = x'$ for some $g \in G$. We let $[x] = xG$. Let

$$X/G := \{[x] : x \in X\}$$

with the quotient topology. Let

$$\pi : X \to X/G$$

be given by $\pi(x) = [x]$.

Every G-space determines a bundle

$$(X, \pi, X/G)$$

and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow^{\pi} & & \downarrow^{\pi} \\ X/G & \xrightarrow{\widetilde{h}} & Y/G \end{array}$$

A bundle (X, p, B) is called a *G-bundle* if X is a G-space and there exists a homeomorphism $f : X/G \to B$ so that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ \downarrow^{\pi} & & \downarrow^{p} \\ X/G & \xrightarrow{f} & B \end{array}$$

A G-bundle for which X is a principal G-space is called a *principal G-bundle*.

Example 2.3. *The previous two examples are principal G-bundles with $X = B \times G$, $X/G \cong B$ in the first example, and in the second example $B = G \backslash \Gamma$.*

2.2. **Numerable Principal Bundles.** Let B be a topological space. Let $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$ be an open cover of B. We say $\{U_i\}_{i \in \mathbb{N}}$ is *numerable* if there exists a partition of unity subordinate to $\mathcal{U}$. (Note: if B is paracompact, this is possible for any open cover of B.)

A principal G-bundle $\xi = (X, p, B)$ is *numerable* if there is a numerable cover $\{U_i\}_{i \in \mathbb{N}}$ of B such that $\xi|_{U_i}$ is trivial for each $i \in \mathbb{N}$.

2.3. **The Functor k_G.** Let B be a topological space, G be a topological group. Let

$$k_G(B) = \left\{ \begin{array}{c} \text{isomorphism classes of numerable} \\ \text{principal } G\text{-bundles over } B \end{array} \right\}.$$

If $f : A \to B$ is continuous, then

$$k_G(f) : k_G(B) \to k_G(A)$$

is defined by

$$k_G(f)[\xi] = [f^*\xi].$$

Proposition 2.4. *If $f : A \to B$ is a homotopy equivalence, then $k_G(f)$ is a bijection.*

Corollary 2.5. *If B is contractible, then every numerable principal G-bundle over B is trivial.*

Note that k_G is a covariant functor from the category of topological spaces to the category of sets.

Definition 2.6. *Fix a numerable principal G-bundle $\omega = (E_0, p_0, B_0)$, and define a map*

$$\phi_w : [\cdot\ , B_0] \to k_G(\cdot)$$

by pullback. If it happens that ϕ_w is an isomorphism, we call ω a universal G-bundle and B_0 a classifying space. *Note that "ϕ_w isomorphism" means that $\phi_w(Z)$ is a bijection for each Z.*

3. Another approach to Classifying Spaces

Let G be a topological group that is a CW complex with a countable number of cells, endowed with cellular multiplication and inverse.

A universal bundle for G is a connected principal G-bundle EG such that $\pi_n(EG) = 0$ for $n > 0$. We define

$$BG := EG \diagup G,$$

so that $\pi_1(BG) = \pi_0(G)$ (from homotopy exact sequence of fiber bundle).

Example 3.1. *$G = \{e\}$, $EG =$ any contractible CW complex, $BG \cong EG$.*

Example 3.2. *$G = \mathbb{Z}$, $EG = \mathbb{R}$, $BG = S^1$.*

Example 3.3. *$G = U(n)$, $EU(n) =$ space of orthonormal $n-frames$ in a complex separable infinite-dimensional Hilbert space $\mathcal{H}$. $BU(n) = \{$subspaces V of $\mathcal{H}$: $\dim V = n\}$.*

In general, what is it?

Definition 3.4. *The* infinite join *EG of G. Consider the set*

$$G * G * ... = \left\{ \sum_{k=0}^{\infty} t_k g_k : \text{all but finitely many of the } t_j \text{ are zero}, \ \sum t_k = 1, g_k \in G \right\}.$$

We define

$$EG = G * G * ... \diagup \tilde{}\,,$$

where

$$\sum_{k=0}^{\infty} t_k g_k \tilde{} \sum_{k=0}^{\infty} t'_k g'_k$$

if $t_k = t'_k$ for all k, $g_k = g'_k$ whenever $t_k, t'_k \neq 0$.

You can check that G acts on EG on the right in the obvious way. The topology on EG is defined as follows. For each m, we have functions

$$\tau_m : EG \to [0,1],$$
$$\tau_m\left(\sum_{k=0}^{\infty} t_k g_k\right) = t_m;$$
$$\gamma_m : \tau_m^{-1}[0,1] \to G,$$
$$\gamma_m\left(\sum_{k=0}^{\infty} t_k g_k\right) = g_m.$$

The topology on EG is the smallest (weakest) that makes these functions continuous.

3.1. **Joins.** Let X and Y be topological spaces. As before,

$$X * Y = \{tx + (1-t)\,y : t \in [0,1], x \in X, y \in Y\}$$

(set of formal linear combinations, with equivalence relation $1x + 0y_1 = 1x + 0y_2$.) We have maps

$$\begin{aligned} \tau_X, \tau_Y &: X * Y \to [0,1] \\ \tau_X(tx + (1-t)\,y) &= t \\ \tau_Y(tx + (1-t)\,y) &= 1-t \end{aligned}$$

and

$$\begin{aligned} \gamma_X &: \tau_X^{-1}((0,1]) \to X \\ \gamma_Y &: \tau_Y^{-1}([0,1)) \to Y \\ \gamma_X(tx + (1-t)\,y) &= x \\ \gamma_Y(tx + (1-t)\,y) &= y \end{aligned}$$

The topology on $X * Y$ is the weakest topology that makes all of these functions continuous. This is not necessarily the same as the quotient topology.

Example 3.5. *$(0,1) * \{pt\}$ is the set of lines, which is an open triangle (one solid open side). The subspace topology in the plane is the join topology. This is not the quotient topology, which is the quotient of $(0,1) \times [0,1]$ mod quotient $(0,1) \times \{1\} = \{pt\}$. Then a set that is an open set containing points of $(0,1) \times \{1\}$ upstairs is not open in the subspace topology.*

Proposition 3.6. *For all $r \in \mathbb{N}$,*

$$\widetilde{H}_{r+1}(X * Y) = \sum_{i+j=r} \widetilde{H}_i(X) \otimes \widetilde{H}_j(Y) \oplus \sum_{i+j=r-1} \operatorname{Tor}\left(\widetilde{H}_i(X), \widetilde{H}_j(Y)\right).$$

Proof. Define

$$\begin{aligned} \widetilde{X} &= \left\{tx + (1-t)\,y : t \le \frac{1}{2}\right\} \\ \widetilde{Y} &= \left\{tx + (1-t)\,y : t \ge \frac{1}{2}\right\} \end{aligned}$$

Identify

$$\begin{aligned} X &\leftrightarrow X * Y \text{ at } t=0 \\ Y &\leftrightarrow X * Y \text{ at } t=1 \\ X * Y &\leftrightarrow X * Y \text{ at } t=\frac{1}{2} \end{aligned}$$

Then X is a deformation retract of $\widetilde{X}$, Y is a deformation retract of $\widetilde{Y}$, $X \times Y \cong \widetilde{X} \wedge \widetilde{Y}$.
reduced Mayer-Vietoris:

$$\ldots \xrightarrow{\phi} \widetilde{H}_{r+1}(X * Y) \xrightarrow{\partial} \widetilde{H}_r\left(\widetilde{X} \wedge \widetilde{Y}\right) \xrightarrow{\psi} \widetilde{H}_r\left(\widetilde{X}\right) \otimes \widetilde{H}_r\left(\widetilde{Y}\right) \xrightarrow{\phi} \widetilde{H}_r(X * Y) \xrightarrow{\partial} \ldots$$

or with the identifications,

$$\ldots \xrightarrow{\phi} \widetilde{H}_{r+1}(X * Y) \xrightarrow{\partial} \widetilde{H}_r(X \times Y) \xrightarrow{\psi} \widetilde{H}_r(X) \otimes \widetilde{H}_r(Y) \xrightarrow{\phi} \widetilde{H}_r(X * Y) \xrightarrow{\partial} \ldots$$

Claim: the inclusions $j_X : X \hookrightarrow X * Y$ and $j_Y : Y \hookrightarrow X * Y$ are homotopic to constant maps. Proof for j_Y : Fix $x_0 \in X$, $y \mapsto 0x_0 + (1-0)\,y$, then $\{tx_0 + (1-t)\,y\}$ is a homotopy from j_Y to the constant function x_0.
Claim: This implies ϕ maps are zero. Thus ψ maps are onto. This gives us

$$0 \to \widetilde{H}_{r+1}(X * Y) \xrightarrow{\partial} \widetilde{H}_r(X \times Y) \xrightarrow{\psi} \widetilde{H}_r(X) \otimes \widetilde{H}_r(Y) \to 0.$$

Using the Künneth Formula:

$$H_r(X \times Y) \cong \sum_{i+j=r} H_i(X) \otimes H_j(Y) \oplus \sum_{i+j=r-1} \operatorname{Tor}\left(H_i(X), H_j(Y)\right).$$

Plug this into the middle term. Then it is *easy* to see that $\ker\psi$ is the stated formula. □

Example 3.7. $r=2$*:*

$$\begin{aligned} H_2(X \times Y) &\cong \sum_{i+j=2} H_i(X) \otimes H_j(Y) \oplus \sum_{i+j=1} \operatorname{Tor}\left(H_i(X), H_j(Y)\right) \\ &\cong \widetilde{H}_2(X) \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \widetilde{H}_2(Y) \oplus \sum_{i+j=2} \widetilde{H}_i(X) \otimes \widetilde{H}_j(Y) \\ &\quad \oplus \sum_{i+j=1} \operatorname{Tor}\left(\widetilde{H}_i(X), \widetilde{H}_j(Y)\right) \oplus \operatorname{Tor}\left(\widetilde{H}_1(X), \mathbb{Z}\right) \oplus \operatorname{Tor}\left(\mathbb{Z}, \widetilde{H}_1(Y)\right) \\ &\cong \widetilde{H}_2(X) \oplus \widetilde{H}_2(Y) \oplus \sum_{i+j=2} \widetilde{H}_i(X) \otimes \widetilde{H}_j(Y) \oplus \sum_{i+j=1} \operatorname{Tor}\left(\widetilde{H}_i(X), \widetilde{H}_j(Y)\right) \\ &\cong \widetilde{H}_2(X) \oplus \widetilde{H}_2(Y) \oplus \ker\psi \end{aligned}$$

Thus,

$$\widetilde{H}_3(X * Y) = H_3(X * Y) \cong \sum_{i+j=2} \widetilde{H}_i(X) \otimes \widetilde{H}_j(Y) \oplus \sum_{i+j=1} \operatorname{Tor}\left(\widetilde{H}_i(X), \widetilde{H}_j(Y)\right).$$

Proposition 3.8. *If Y is arc-connected (injective-path-connected) and if X is nonempty, then $X * Y$ is simply connected.*

Proof. Suppose $f : S^1 \to X * Y$ is continuous. Write

$$f(z) = t(z)\,\alpha(z) + (1 - t(z))\,\beta(z),$$

where $\alpha(z)$ is defined when $t(z) \neq 0$, $\beta(z)$ is defined when $t(z) \neq 1$. We extend $\beta' : S^1 \to Y$ so that $\beta'(z) = \beta(z)$ when $t \leq \frac{1}{2}$. (Just connect end points of the image $\beta\left(\left[0, \frac{1}{2}\right]\right)$.) Now, fix $x_0 \in X$. Then you write down a homotopy of f to a constant (x_0) by using x_0, β' and α. ☐

Proposition 3.9. *Let $X_0, ..., X_n$ be nonempty, and suppose that X_i is $(c_i - 1)$-connected for $0 \leq i \leq n$. Then*

$$X_0 * X_1 * ... * X_n$$

is $(c_0 + c_1 + ... + c_n + n - 1)$-connected.

Proof. Note that -1-connected means nonempty. Enough to show for $n = 1$. Case 1: $c_0 = c_1 = 0$, because the join of any two nonempty spaces is connected. Case 2: if either c_0 or c_1 is > 0, by the previous proposition, $X_0 * X_1$ is 1-connected. This implies

$$H_r(X_0 * X_1) = 0$$

for $r \leq c_0 + c_1$. This implies $X_1 * X_0$ is $c_0 + c_1 = c_0 + c_1 + 1 - 1$ connected. ☐

Corollary 3.10. *The join of any $n + 1$ nonempty spaces is $(n - 1)$-connected.*

Corollary 3.11. *$G * G * G * ...$ is ∞-connected.*

The following theorem is a corollary of the last corollary.

Theorem 3.12. *EG is a universal bundle for G.*

4. Algebraic K-theory

Let R be a ring with unit. Then let $K_0(R)$ be the Grothendieck completion of the abelian monoid of isomorphism classes of finitely generated projective modules (direct summands of free modules) over R. Everything in $K_0(R)$ is equivalent to $[p] - [R^n]$, with p projective.

Observe

$$\begin{array}{rcl} GL(n,R) & \hookrightarrow & GL(n+1,R) \\ S & \mapsto & \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Let

$$GL(R) = \varinjlim GL(n,R).$$

For each $n \in \mathbb{N}$, $1 \leq i, j \leq n$, $i \neq j$, $a \in R$, let $e_{ij}(a) \in GL(n,R) \hookrightarrow GL(R)$ be defined as the identity matrix except with a placed in the (i,j)-entry. Note that $e_{ij}(a)^{-1} = e_{ij}(-a)$. Let $E(n,R)$ be the subgroup of $GL(n,R)$ generated by $\{e_{ij}(a) : i \neq j, a \in R\}$. Let

$$E(R) = \varinjlim E(n,R).$$

Proposition 4.1. *The group $E(n,R)$ is a normal subgroup of $GL(n,R)$, and $E(R)$ is a normal subgroup of $GL(R)$. In fact, $E(R)$ is the commutator subgroup of $GL(R)$.*

Properties of $E(R)$:

- $[GL(R), GL(R)] = E(R)$ (see above).

- $[E(R), E(R)] = E(R)$
- $GL(R) \diagup E(R)$ is abelian.

What is in $E(R)$?

- any upper- or lower- triangular matrix with 1s on the diagonal.
- (Whitehead Lemma) For any matrix $A \in GL(R)$,

$$\begin{pmatrix} A & 0 & & & 0 \\ 0 & A^{-1} & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \in E(R)$$

What's not in $E(R)$?

- anything with $\det \neq 1$. (sometimes these are the only examples).

We define the algebraic K-group K_1 by

$$K_1(R) := GL(R) \diagup E(R).$$

Because of the commutator property, it is abelian — ie the abelianization of $GL(R)$.

Then we define the Steinberg group as follows. Let $St(n, R)$ be the group of generated by $\{x_{ij}(a) : i \neq j, a \in R, 1 \leq i, j \leq n\}$ with relations

$$\begin{aligned} x_{ij}(a)\, x_{ij}(b) &= x_{ij}(a+b) \\ x_{ij}(a)\, x_{kl}(b) &= x_{kl}(b)\, x_{ij}(a),\ j \neq k, i \neq l, \\ & [x_{ij}(a), x_{jk}(b)] = x_{ik}(ab), i, j, k \text{ distinct.} \end{aligned}$$

(same trivial relations that elementary relations satisfy)

There is a map

$$St(n, R) \to St(n+1, R)$$

(not necessarily injective). We set

$$St(R) = \lim_{\rightarrow} St(n, R).$$

Therefore there is a group homomorphism

$$\begin{aligned} \phi &: St(R) \twoheadrightarrow E(R) \\ \phi(x_{ij}(a)) &= e_{ij}(a). \end{aligned}$$

The kernel is

$$K_2(R) = \ker \phi$$

Theorem 4.2. *$K_2(R)$ is abelian.*

Fun way to get elements of $K_2(R)$ when R is commutative: given $A, B \in E(R)$, choose $a, b \in St(R)$ such that $\phi(a) = A$, $\phi(b) = B$. Then $[a, b] \in K_2(R)$, because

$$\phi\left(aba^{-1}b^{-1}\right) = ABA^{-1}B^{-1} = I.$$

But how do we know that $aba^{-1}b^{-1}$ is nontrivial in $St(R)$?

Fun way to get elements of $E(R)$ when R is commutative: Take $r, s \in R^* = \{\text{units}\}$. Then

$$\begin{pmatrix} r & 0 & 0 \\ 0 & r^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^{-1} \end{pmatrix} \in E(3, R) \subset E(R).$$

Use previous fun construction above to get an element $\{r, s\}$ of $K_2(R)$. This is called the *Steinberg symbol* of r and s.

Fun Facts about Steinberg symbols:

- If F is a field, $K_2(F)$ is generated by Steinberg symbols.
- $\{-1, -1\}$ is the generator of $K_2(\mathbb{Z})$ and of $K_2(\mathbb{R})$ but is trivial in $K_2(\mathbb{C})$.

Examples: $K_2(\mathbb{Z}) = \mathbb{Z}_2$.

We get an exact sequence:

$$0 \to K_2(R) \to St(R) \to E(R) \to 0$$

In other words, $K_2(R)$ a central extension of $E(R)$ and is "universal".

This group $K_2(R)$ measures the extent to which the elementary matrices have nontrivial relations.

Suppose that I is an ideal in R. Let $D(R, I)$ (the double) be

$$D(R, I) = \{(r_1, r_2) : r_1 - r_2 \in I\}.$$

Define

$$\mu_1, \mu_2 : D(R, I) \to R$$

by

$$\mu_1(r_1, r_2) = r_1, \mu_2(r_1, r_2) = r_2.$$

We get a group homomorphism

$$\mu_{2*} : K_i(D(R, I)) \to K_i(R).$$

Define the relative group

$$K_i(R, I) := \ker(\mu_{2*})$$

The long exact sequence is

$$\begin{aligned} K_2(R, I) &\to K_2(R) \to K_2(R/I) \xrightarrow{\partial} K_1(R, I) \to \\ &\to K_1(R) \to K_1(R/I) \xrightarrow{\partial} K_0(R, I) \to \\ &\to K_0(R) \to K_0(R/I) \end{aligned}$$

Note that this K-theory does not satisfy excision.

5. Central Extensions

Let G be a group, A an abelian group. A central extension of G by A is a short exact sequence

$$0 \to A \to E \xrightarrow{\phi} G \to 0$$

Note that central extensions E and E' are equivalent if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & E & \xrightarrow{\phi} & G & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & A & \to & E' & \xrightarrow{\phi'} & G & \to & 0 \end{array}$$

Alternate picture: start with G, and find homomorphisms $\phi : E \twoheadrightarrow G$ with abelian kernel. A central extension E of G is *universal* if for any other central extension E' of G, there exists a unique homomorphism $\psi : E \to E'$ such that

$$\begin{array}{ccc} E & \overset{\phi}{\to} & G \\ \downarrow^{\psi} & & \| \\ E' & \overset{\phi'}{\to} & G \end{array}$$

commutes.

Proposition 5.1. *G has a universal central extension iff $[G,G] = G$ (ie G is* perfect*).*

Theorem 5.2. *The exact sequence*

$$0 \to K_2(R) \to St(R) \to E(R) \to 0$$

is a universal central extension.

6. Group Homology

Let G be a discrete group. For each $n \in \{0,1,2,...\}$, let

$$\overline{C_n}(G) = \{[g_1|g_2|...|g_n] : g_1, g_2, ..., g_n \in G\}$$

(free G-module generated by these formal objects). The differential $\partial : \overline{C_n}(G) \to \overline{C_{n-1}}(G)$ is defined by

$$\partial = \sum_{j=0}^{n} (-1)^j d_j,$$

where

$$d_j[g_1|g_2|...|g_n] = \begin{cases} g_1[g_2|g_3|...|g_n] & \text{if } j = 0 \\ [g_1|...g_{j-1}|g_j g_{j+1}|g_{j+2}|...|g_n] & \text{if } 0 < j < n \\ [g_1|g_2|...|g_{n-1}] & \text{if } j = n \end{cases}$$

Then $\partial^2 = 0$, and let $H_*(G)$ be the homology of this complex.

Then $H_1(G) = G/[G,G]$.

Proposition 6.1. *Let R be a ring. Let F be the free group on generators $\{x_{ij}(a) : a \in R, i \neq j\} \neq St(R)$. Let N be the normal subgroup of F such that $F/N \cong E(R)$. Then*

$$St(R) = [F,F]/[F,N],$$

and

$$K_2(R) \cong (N \cap [F,F])/[F,N].$$

Theorem 6.2. *We have*

$$K_2(R) \cong H_2(E(R)).$$

What is the isomorphism? Define

$$D : F \to \overline{C_2}(E(R))$$

by

$$D(f) = \sum_{x \in F} \left[\frac{\partial f}{\partial x} \middle| x \right],$$

where the RHS is interpreted bilinearly.

Then D induces the isomorphism on $K_2(R) \cong (N \cap [F,F]) \diagup [F,N]$ to get an element of $H_2(E(R))$.

Example 6.3. *Let*

$$f = x_{12}(a)^2 x_{31}(b)^3 x_{12}(a)^{-1}.$$

Then

$$\begin{aligned} D(f) &= 2\left[x_{12}(a) x_{31}(b)^3 x_{12}(a)^{-1}\middle| x_{12}(a)\right] - \left[x_{12}(a)^2 x_{31}(b)^3 x_{12}(a)^{-2}\middle| x_{12}(a)\right] \\ &\quad +3\left[x_{12}(a)^2 x_{31}(b)^2 x_{12}(a)^{-1}\middle| x_{31}(b)\right]. \end{aligned}$$

Theorem 6.4. *We have $H_*(G) = H_*(K(G,1))$.*

Note one could define $K_3(R) := H_3(St(R))$.

7. Technical Details

7.1. **Homology with local coefficients, homology of covering spaces.** The point: looking at $H_*\left(\widetilde{X}; G\right)$ can be easier and doesn't really lose any information. If $\widetilde{X}$ corresponds to $\pi_1(X) \diagup H$, then we want G to be a $\pi_1(X) \diagup H$-module.

Idea of local coefficients: every simplex has coefficients in some G, but you can sometimes make things more complicated by letting the boundary map take the coefficient group into another coefficient group by a group isomorphism. Also, a local coefficient system is a $\pi_1(X)$-equivariant G-bundle over $\widetilde{X}$. We call this $\mathcal{G}$. We define $H_*(X;\mathcal{G}) = H_*\left(C_*\left(\widetilde{X}\right) \otimes_{\pi_1(X)} G\right)$.

If G is a $\pi_1(X) \diagup H$-module. Then you can let $\widetilde{X}_{\pi_1(X)\diagup H}$ be the intermediate covering space with $\pi_1(X) \diagup H$ as deck transformations. Then

$$H_*(X;\mathcal{G}) = H_*\left(C_*\left(\widetilde{X}_{\pi_1(X)\diagup H}\right) \otimes_{\pi_1(X)\diagup H} G\right).$$

In particular, if we give G the trivial action,

$$H_*\left(C_*\left(\widetilde{X}\right) \otimes_{\pi_1(X)} G\right) = H_*\left(C_*\left(\widetilde{X}_{\pi_1(X)\diagup H}\right) \otimes_{\pi_1(X)\diagup H} G\right) = H_*(X;G).$$

7.2. **Classifying spaces – baby obstruction theory.** Given a group G, we get a space BG, a connected CW complex such that BG is a $K(G,1)$. In particular, it is homotopy equivalent to $K(G,1)$ — ie unique up to homotopy equivalence.

Lemma 7.1. *Suppose X is a CW complex, and $h : \pi_1(X) \to G$ is a homomorphism. Then there is a unique (up to homotopy) map $\overline{h} : X \to BG$ such that $\overline{h} = h : \pi_1(X) \to \pi_1(BG) = G$.*

(Uniqueness of BG follows from this.)

Proof. Assume that X is connected. Assume that X^0 (the zero skeleton) is a point. Map each X^k individually to BG, as follows. Define $\overline{h_k} : X^k \to BG$ by

$$\overline{h_0} : \{pt\} \to \{pt\} = BG^0,$$

$$\overline{h_1} : X^1 = \{loops \in \pi_1(X)\} \to BG, \text{by}$$

$$loop\ e \to loop\ h([e])$$

For $\overline{h_2}$: suppose e^2 is a 2-cell in X. We already know $\overline{h}(\partial e^2)$ has to be a loop in BG, but it is null homotopic, so $[\partial e^2] = 0 \in \pi_1(X)$, so $\left[\overline{h}(\partial e^2)\right] = 0 \in \pi_1(BG)$. So there exists an imbedding $D^2 \to BG$ that represents the collapse. We define $\overline{h_2}(e^2)$ to be that collapsing disk. We continue the same process. By transfinite induction, there exists such a map h. Why is this unique? Suppose there are two such maps. Then we get a map $X \times \{0\} \cup \{1\} \to BG$. Collapse base points on top and bottom of the cylinder $X \times I$ to a point. Again there is no obstruction to extending the map to all of $X \times I$. Thus we get a homotopy. □

Corollaries:

Corollary 7.2. *Given a homomorphism $\phi : H \to G$, there exists a unique map $BH \to BG$, inducing $\phi : \pi_1(BH) \to \pi_1(BG)$.*

Corollary 7.3. *Given a short exact sequence (ie H is normal)*

$$1 \to H \to G \to G/H \to 1$$

there is a fibration

$$BH \to BG \to B(G/H)$$

Proof. From the last corollary, there exists $BG \to B(G/H)$ inducing the right homomorphism on π_1. Let F be its homotopy fiber. So we have a fibration $F \to BG \to B(G/H)$, so by the long exact sequence of π_*, $\pi_1(F) = H$, $\pi_0(F) = 0$, $\pi_k(F) = 0$, $k > 1$. So $F\tilde{}_{h.e.}BH$. □

8. Constructing K-groups using the Plus Construction

Summary of what we know: Basic objects R is a unital ring, $GL(R)$, $E(R)$-elementary matrices, $St(R) = \{x_{ij}(a), i \neq j, a \in R \mid$ "obvious" elem matrix relations$\}$ - Steinberg group, ideal $I \subset R$.

$$\begin{aligned} K_0(R) &= \text{Grothendieck group of projective } R \text{ modules} \\ K_0(R, I) &= \ker(p_{1*} : K_0(D(R, I)) \to K_0(R)) \text{ where} \\ D(R, I) &= \{(x, y) \in R \times R : x - y \in I\} \end{aligned}$$

$$\begin{aligned} K_1(R) &= GL(R)/E(R) = GL(R)_{ab} = GL(R)/[GL(R), GL(R)] \\ E(R) &= [E(R), E(R)] \ (E(R) \text{ is perfect}) \\ &= [GL(R), GL(R)] \\ K_1(R, I) &= \ker(p_{1*} : K_1(D(R, I)) \to K_1(R)) \end{aligned}$$

$$\begin{aligned} K_2(R) &= \ker(St(R) \to E(R)) = H_2(E(R)) = H_2(BE(R)) \\ &\quad \text{by (definition,hard,group homology theory)} \end{aligned}$$

$$\begin{aligned} &St(R) \text{ is a is the universal central extension of } E(R) \text{ by} \\ 1 \to\ &K_2(R) \to St(R) \to E(R) \to 1 \end{aligned}$$

There is a long exact sequence

$$K_2(R) \to K_2(R/I) \to K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(R, I) \to K_0(R) \to K_0(R/I)$$

Suppose we have a space X, and let π is a perfect subgroup of $\pi_1(X)$. Suppose we have a construction $X \to X^+$ such that

(1) $\pi_1(X^+) = \pi_1(X)/\pi$

(2) $H_*(X^+) \cong H_*(X)$
(3) Suppose that if $E \to X$ is a covering map and $\pi < \pi_1(E)$, then we obtain a covering map $E^+ \to X^+$.

This is all we need to get a series of spaces representing K-theory, because ....

Notice that $E(R)$ is perfect in $GL(R) = \pi_1(BGL(R))$. So we can form $BGL(R)^+$. Also, $E(R)$ is perfect in $E(R) = \pi_1(BE(R))$, so we have a covering $BE(R)^+ \to BGL(R)^+$. The space $BGL(R)$ has a cover corresponding to $E(R)$, but all the higher homotopy of that vanish except π_1, so that cover has to be $BE(R)$. So we can make these constructions.

We will now show that $\pi_1\left(BGL(R)^+\right) = K_1(R)$, $\pi_2\left(BGL(R)^+\right) = K_2(R)$. Observe that

$$\begin{aligned}\pi_1\left(BGL(R)^+\right) &= \pi_1(BGL(R)) \diagup E(R) \\ &= GL(R) \diagup E(R) = K_1(R).\end{aligned}$$

Next,

$$\pi_2\left(BGL(R)^+\right) = \pi_2\left(BE(R)^+\right)$$

Since $BE(R)^+$ is simply connected by property (1) above. Then

$$\begin{aligned}\pi_2\left(BGL(R)^+\right) &= H_2\left(BE(R)^+\right) \text{ (Whitehead)} \\ &= H_2(BE(R)), \text{ so by property (2)} \\ &= H_2(E(R)) = K_2(R).\end{aligned}$$

We define

$$\mathbf{K}(R) := BGL(R)^+ \times K_0(R)$$

Since $K_0(R)$ is discrete,

$$\begin{aligned}\pi_0(\mathbf{K}(R)) &= K_0(R), \\ \pi_i(\mathbf{K}(R)) &= \pi_i\left(BGL(R)^+\right) =: K_i(R), i > 0\end{aligned}$$

Yay. So if the plus construction $X \to X^+$, we can define the higher algebraic K groups.

9. The Plus Construction

Theorem 9.1. *Let X be a connected CW complex with base point x_0. Let π be a perfect normal subgroup of $\pi_1(X)$. Then there is a space X^+, obtained by attaching 2- and 3-cells to X such that*

(1) *$\pi_1(X) \to \pi_1(X^+)$ is the quotient $\pi_1(X) \diagup \pi$.*
(2) *$H_*(X; M) \to H_*(X^+; M)$ is an isomorphism for any $\pi_1(X) \diagup \pi$ module M.*

Furthermore, X^+ is unique in the sense that any other such space containing X is homotopy equivalent relative to X.

Proof. Let $\{g_i\}$ be a set of generators for π. (note that they are homologically trivial due to abelianization). For each g_i, attach a 2-cell at each loop g_i. The result is $Y = X \cup_{\{g_i\}} (\text{2-cell})$. Note that $\pi_1(Y) = \pi_1(X) \diagup \pi$. Note that from the inclusion $X \to Y$, we have

$$\begin{array}{ccc}\widetilde{X} & \to & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \to & Y\end{array}$$

where $\widetilde{X}$ is the cover with deck transformations $\pi_1 \diagup \pi$, and $\widetilde{Y}$ is the (universal) cover with deck transformations $\pi_1 \diagup \pi$. Then $\pi_1\left(\widetilde{X}\right) = \pi$, $H_1\left(\widetilde{Y}\right) = 0$, and $H_1\left(\widetilde{X}\right) = \pi_{ab} = 0$. Then

$H_2(Y,X) = \bigoplus_{g_i} \mathbb{Z}$ (free abelian on added 2-cells). Then $H_2\left(\widetilde{Y},\widetilde{X}\right) = \bigoplus_{g_i} \mathbb{Z}\left[\pi_1 \diagup \pi\right]$, the free $\mathbb{Z}\left[\pi_1 \diagup \pi\right]$-module gen by the lifts of the 2-cells. In general, $H_*\left(\widetilde{Y},\widetilde{X}\right) = H_*(Y,X) = 0$ for $* \neq 2$. then from

$$\to H_2\left(\widetilde{X}\right) \to H_2\left(\widetilde{Y}\right) \to H_2\left(\widetilde{Y},\widetilde{X}\right) \to H_1\left(\widetilde{X}\right) \to$$

Since $H_2\left(\widetilde{Y},\widetilde{X}\right)$ is free, the map $H_2\left(\widetilde{Y}\right) \to H_2\left(\widetilde{Y},\widetilde{X}\right)$ splits, so you can write

$$H_2\left(\widetilde{Y}\right) \cong A \oplus H_2\left(\widetilde{Y},\widetilde{X}\right)$$

Important part: Since $\pi_1\left(\widetilde{Y}\right) = 0$, by the Whitehead-Hurewicz,

$$\pi_2\left(\widetilde{Y}\right) = H_2\left(\widetilde{Y}\right) \supset H_2\left(\widetilde{Y},\widetilde{X}\right)$$

Thus, every element of $H_2\left(\widetilde{Y},\widetilde{X}\right)$ can be represented by a 2-sphere. Now take each S^2 representing a generator of $H_2\left(\widetilde{Y},\widetilde{X}\right)$, then project into Y, attach a 3-cell to each such sphere, and call the result X^+. Notice that $\widetilde{X^+}$ is what we get by doing the same thing upstairs.

Next,

(1) $\pi_1(X^+) = \pi_1(Y) = \pi_1(X) \diagup \pi$ because adding 3-cells doesn't affect π.
(2) $H_*(Y,X;M) = H_*\left(C_*\left(\widetilde{Y},\widetilde{X}\right) \otimes_{\mathbb{Z}[\pi_1 \diagup \pi]} M\right) = \bigoplus_{g_i} M$ (in dim 2 and zero otherwise) is freely generated by the attached 2-cells. Similarly, $H_*(X^+,X;M) = \bigoplus_{g_i} M$ (in dim 3 and zero otherwise). Now, consider the long exact sequence of the triple (X^+,Y,X); thus we get an isomorphism between the only two nontrivial parts:

$$H_3\left(X^+,Y\right) \to H_2(Y,X)$$

Thus, $H_*(X^+,X) = 0$, so $H_*(X) \cong H_*(X^+)$. (suppressing M notation).

To show uniqueness, assume we have X^+ and Z^+ containing X with all the properties. We now need to obtain a map $X^+ \to Z^+$. Note that the image of g_i in Z^+ is homotopically trivial in X_1^+, so we extend the identity to $Y \to Z^+$. We need to extend this map to X^+. This is sufficient, because if we can do the extension, then we have $X^+ \to Z^+$ induces a map on $\pi_1(\cdot)$, which will be an isomorphism by construction. Now we can lift to universl covers, and you have a map between simply connected spaces, and the homology $H_*(X^+) \to H_*(Z^+)$ is an isomorphism through $H_*(X^+) \to H_*(X) \to H_*(Z^+)$. By the Whitehead theorem, the universal covers of X^+ and Z^+ are homotopy equivalent, so the homotopy extends to the base spaces. Thus, it is sufficient to extend the map $Y \to Z^+$ extends to X^+. We just need to show that the image of the attaching 2-spheres is trivial in $\pi_2(Z^+)$. But

$$\begin{aligned} \pi_2\left(Z^+\right) &= \pi_2\left(\widetilde{Z^+}\right) = H_2\left(\widetilde{Z^+}\right) \\ &= H_2\left(\widetilde{X}\right) \text{ by property 2} \end{aligned}$$

Thus the attaching sphere is null homotopic in Z^+ iff it is hull homologous in $H_2\left(\widetilde{X}\right) \cong H_2\left(\widetilde{X^+}\right)$. But our attaching cells are zero in $H_2\left(\widetilde{X^+}\right)$ because they are attaching cells. $\square$

Then, observe that

$$\begin{aligned} K(R) &= BGL(R)^+ \times K_0(R), \\ \pi_j(K(R)) &= K_j(R) \text{ for } j = 0, 1, 2. \end{aligned}$$

Then

$$\pi_3\left(BG^+\right) = H_3(\widehat{\pi}),$$

where $\widehat{\pi}$ is the universal central extension of π (which exists since π is perfect). Then

$$\begin{aligned} K_3(R) &= H_3(St(R)) \\ \pi_j\left(BG^+\right) &= \pi_j\left(B\widehat{\pi}^+\right) \text{ for } j \geq 3. \end{aligned}$$

10. Higher K-theory

We have been looking at

$$K_i(R) = \pi_i\left(BGL(R)^+\right), \ i > 0.$$

Let G be any group ($GL(R)$ in our case). Let π be the commutator subgroup, which we assume to be perfect (so $\pi = [G, G] = [\pi, \pi]$) ($E(R)$ in our case). Let $\widehat{\pi}$ be the universal central extension of π ($St(R)$ in our case). We have the covering $B\pi^+ \to BG^+$, which implies that $\pi_j(BG^+) = \pi_j(B\pi^+)$, $j \geq 2$, and in fact $\pi_1(BG^+) = G_{ab}$, $\pi_2(BG^+) = H_2(\pi)$. Next, we will see that

$$\begin{aligned} \pi_3\left(BG^+\right) &= H_3(\widehat{\pi}, \mathbb{Z}) \\ \pi_j\left(BG^+\right) &= \pi_j\left(B\widehat{\pi}^+\right) \end{aligned}$$

A consequence,

$$\begin{aligned} K_3(R) &= H_3(St(R), \mathbb{Z}) \\ K_j(R) &= \pi_j\left(BSt(R)^+\right). \end{aligned}$$

Group homology:

(1) G has a universal central extension iff G is perfect.
(2) If E is the universal central extension (UCE), then E is perfect, and all central extensions of E are trivial.
(3) If E is an UCE of G, then we have a short exact sequence

$$1 \to H_2(G, \mathbb{Z}) \to E \to G \to 1,$$

and $H_2(E) = 0$.

We look at $\pi_j(B\pi^+)$ ($= \pi_j(BG^+)$).

We have the UCE diagram

$$1 \to H_2(\pi, \mathbb{Z}) \to \widehat{\pi} \to \pi \to 1,$$

which implies the fibration

$$BH_2(\pi, \mathbb{Z}) \to B\widehat{\pi} \xrightarrow{Bq} B\pi.$$

Since $\widehat{\pi}$ is perfect, there exists $B\widehat{\pi}^+$. We get a map

$$B\widehat{\pi}^+ \to B\pi^+.$$

Now we have the diagram

$$\begin{array}{ccccc} BH_2(\pi,\mathbb{Z}) & \to & B\widehat{\pi} & \to & B\pi \\ \downarrow \tau \vdots \sigma \uparrow & & \downarrow f & & \downarrow \\ F & \to & B\widehat{\pi}^+ & \to & B\pi^+ \end{array} \tag{10.1}$$

where F is the homotopy fiber. Note that $H_2(B\widehat{\pi}^+) = H_2(B\widehat{\pi})$ by the plus construction, and since $\widehat{\pi}$ is UCE, $H_2(B\widehat{\pi}^+) = 0 = H_2(\widehat{\pi})$. Also, $\pi_1(B\widehat{\pi}^+) = \widehat{\pi}_{ab} = 0$. Thus, $B\widehat{\pi}^+$ is 2-connected. By Hur.,

$$\pi_3\left(B\widehat{\pi}^+\right) = H_3\left(\widehat{\pi}^+\right) = H_3\left(B\widehat{\pi}\right).$$

So we want to show that

$$\pi_3\left(B\pi^+\right) = \pi_3\left(B\widehat{\pi}^+\right).$$

So the interesting map in the diagram (10.1) is f. In this diagram, the image of $BH_2(\pi,\mathbb{Z})$ is null homotopic in $B\pi^+$. So there is a lift τ. It is "easy to see" from a diagram chase that τ induces an isomorphism on π_1. By baby obstruction theory: there exists a lift $\sigma : F \to BH_2(\pi,\mathbb{Z})$, inducing an inverse. Then $\sigma\tau = \mathbf{1}$. So $BH_2(\pi,\mathbb{Z})$ splits F, so $H_*(F) = H_*(BH_2(\pi,\mathbb{Z})) \oplus X$. Similarly for π_*. It suffices to show that $X = 0$.

Let $E \xrightarrow{q} B$ (generalization of $B\widehat{\pi}^+ \to B\pi^+$) be a fibration, and we restrict the fibration to $E_0 \xrightarrow{q} B_0 = B\pi$. We have (B, B_0) is acyclic ($H_*(B, B_0) = 0$), (E, E_0) and $(E = B\widehat{\pi}^+, B\widehat{\pi})$ is acyclic by the plus construction, so by the long exact sequence of the triple $(E, E_0, B\widehat{\pi})$, $(E_0, B\widehat{\pi})$ is also acyclic. The diagram (10.1) restricts to

$$\begin{array}{ccccc} BH_2(\pi,\mathbb{Z}) & \to & B\widehat{\pi} & \to & B\pi \\ & & \downarrow f, g \uparrow & & || \\ F & \to & E_0 & \to & B\pi \end{array}$$

By the long exact homotopy sequence and the five lemma, we get

$$\pi_1(E_0) = \widehat{\pi}.$$

Again, by baby obstruction theory, there exists the splitting g so that $B\widehat{\pi}$ splits E_0, so

$$\pi_j(E_0) = \pi_j(B\widehat{\pi}) \oplus Y.$$

From the long exact homotopy sequence, $\pi_j(F) \to \pi_j(E_0)$ is an isomorphism for $j > 1$. Thus, $\pi_j\left(\widetilde{F}\right) \to \pi_j\left(\widetilde{E_0}\right)$ is an isomorphism for $j > 1$. Suppose that $\pi_k\left(\widetilde{F}\right) \neq 0$, with k the smallest possible, Then $H_k\left(\widetilde{E_0}\right) \neq 0$. As a result, $B\widehat{\pi} \to E_0$ is not a homology isomorphism in degree k, which is contradicts acyclicity of $(E_0, B\widehat{\pi})$, so we get the h.e. between $B\pi^+$ and $B\widehat{\pi}^+$.

11. Comparison of Algebraic and Topological K-theories

Definition 11.1. *A (unital) C^*-*algebra *is a complex $*$-algebra A equipped with an involution $*$ and a norm $\|\cdot\|$ satisfying the following properties:*

- *A is complete in the metric $d(x,y) = \|x-y\|$;*
- $\|x+y\| \leq \|x\| + \|y\|$ for all x and y in A;
- $\|xy\| \leq \|x\|\,\|y\|$ for all x and y in A;

- $\|x^*x\| = \|x\|^2$ (The C^* condition). (implies but is not implied by $\|x^*\| = \|x\|$.)

Example 11.2. *$B(\mathcal{H})$ = continuous linear maps on a Hilbert space $\mathcal{H}$; norm is the operator norm, and $*$ is the adjoint.*

Example 11.3. *$C(X)$ =complex-valued continuous functions on a compact Hausdorff space X, with sup norm and complex conjugation. Every commutative C^* algebra has this form.*

Theorem 11.4. *There exists at most one norm that makes a complex $*$-algebra into a C^*-algebra, and if A is a C^*-algebra, then $M(n, A)$ is also a C^*-algebra for every natural number n.*

Definition 11.5. *Let x be an element of a C^*-algebra A. The* spectrum *of x is the set*

$$\operatorname{spec}(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \textit{ is not invertible.}\}$$

Note: *spec* (a) is always a compact, nonempty, bounded proper subset of $\mathbb{C}$.

Definition 11.6. *(holomorphic functional calculus) Let A be a C^*-algebra and let x be an element of A. Suppose that Ω is an open set that contains* $\operatorname{spec}(x)$*, let ϕ be an analytic function on Ω, and choose a closed piecewise smooth curve C that surrounds* $\operatorname{spec}(x)$ *and is contained in Ω. Then we define*

$$\phi(x) = \frac{1}{2\pi i} \int_C \phi(z)(x - z1)^{-1}\, dz \in A.$$

This definition does not depend on the choice of Ω or C.

Definition 11.7. *(just for these notes!!) A $*$-subalgebra $\mathcal{A}$ of a C^*-algebra is* smooth *if $M(n, \mathcal{A})$ is closed under the holomorphic functional calculus for every natural number n.*

Example 11.8. *$C^\infty(M)$ for any compact manifold M.*

Throughout the rest of these notes, $\mathcal{A}$ will denote a smooth algebra.

Definition 11.9. *$M(\mathcal{A}) = \varinjlim M(n, \mathcal{A})$.*

Proposition 11.10. *$M(\mathcal{A})$ is smooth.*

Definition 11.11. *$K_0^{top}(\mathcal{A})$ = Grothendieck completion of the abelian monoid of similarity classes of idempotents in $M(\mathcal{A})$ = Grothendieck completion of the abelian monoid of homotopy classes of idempotents in $M(\mathcal{A})$.*

Proposition 11.12. *$K_0^{top}(\mathcal{A}) \cong K_0^{alg}(\mathcal{A})$ = Grothendieck completion of f.g. projective modules in $\mathcal{A}$, for every smooth algebra $\mathcal{A}$.*

Definition 11.13. *For each natural number n, let $GL(n, \mathcal{A})_0$ denote the connected component of the identity matrix. Then we define*

$$GL(\mathcal{A}) = \varinjlim GL(n, \mathcal{A})$$
$$GL(\mathcal{A})_0 = \varinjlim GL(n, \mathcal{A})_0.$$

Proposition 11.14. *$GL(\mathcal{A})_0$ is a normal subgroup of* $\mathrm{GL}(\mathcal{A})$*.*

Definition 11.15. $\mathrm{K}_1^{\mathrm{top}}(\mathcal{A}) = \mathrm{GL}(\mathcal{A}) / \mathrm{GL}(\mathcal{A})_0$.

Note that K_1^{top} is abelian. $\begin{pmatrix} ST & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \sim \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \sim \begin{pmatrix} TS & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition 11.16. *There exists a natural surjective group homomorphism from* $\mathrm{K}_1^{\mathrm{alg}}(\mathcal{A})$ *to* $\mathrm{K}_1^{\mathrm{top}}(\mathcal{A})$.

Proof. $\mathrm{K}_1^{\mathrm{alg}}(\mathcal{A}) = \mathrm{GL}(\mathcal{A})/\mathrm{E}(\mathcal{A})$, and $\mathrm{E}(\mathcal{A}) = [\mathrm{GL}(\mathcal{A}), \mathrm{GL}(\mathcal{A})]$ is a (normal) subgroup of $\mathrm{GL}(\mathcal{A})_0$. □

Note: almost never is this an isomorphism. For example, $K_1^{\mathrm{alg}}(\mathbb{C}) \cong \mathbb{C}^*$, $K_1^{\mathrm{top}}(\mathbb{C}) \cong \{1\}$.

(Why is $K_1^{\mathrm{top}}(\mathbb{C}) = \{1\}$? Suppose $M \in GL(A)$ is a product of exponentials $e^{A_1}...e^{A_n}$, then it is homotopic to the identity. Further, such products generate a subgroup, and you get a open set containing the identity.)

What about the higher K groups? We first need a definition of K-theory for smooth algebra without unit.

11.1. Smooth nonunital algebras.

Suppose that $\mathcal{J}$ is a smooth nonunital algebra. Define

$$\mathcal{J}^+ = \{(x, \lambda) \in \mathcal{J} \oplus \mathbb{C}\}$$

to be an algebra with componentwise addition and scalar multiplication and algebra multiplication given by the formula

$$(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu).$$

(Think of these as $x + \lambda I$ etc.) Then $\mathcal{J}^+$ is a smooth algebra with multiplicative identity $(0, 1)$.

Note that $\mathcal{J}^+/\mathcal{J} \cong \mathbb{C}$. Observe that if Z is locally compact, $C_0(Z)^+ \cong C(Z^+)$, where on the right $+$ denotes the one-point compactification.

Definition 11.17. $K_0^{\mathrm{top}}(\mathcal{J}) = \ker\left(K_0^{\mathrm{top}}(\mathcal{J}^+) \longrightarrow K_0^{\mathrm{top}}(\mathcal{J}^+/\mathcal{J}) = K_0^{\mathrm{top}}(\mathbb{C})\right)$.

Proposition 11.18. *For every smooth unital* $\mathbb{C}$*-algebra* $\mathcal{A}$*, there exists an isomorphism*

$$\Sigma : K_1^{\mathrm{top}}(\mathcal{A}) \longrightarrow K_0^{\mathrm{top}}\left(C_0^\infty(\mathbb{R}, \mathcal{A})\right).$$

Proof. Here's the definition of Σ: identify $\mathbb{R}$ with $(0, 1)$ and take S in $\mathrm{GL}(n, \mathcal{A})$. Then $\mathrm{diag}(S, S^{-1})$ is in $\mathrm{GL}(2n, \mathcal{A})$, in fact in the connected component of I, and there exists an element T in $\mathrm{GL}(2n, C^\infty([0, 1], \mathcal{A})$ such that

$$T(0) = \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \quad \text{and} \quad T(1) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then

$$\Sigma[S] = \left[T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1}\right] - \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Proof is left to the interested reader. It has something to do with

$$0 = K_1^{\mathrm{top}}\left(C_0^\infty([0, 1), \mathcal{A})\right) \to K_1^{\mathrm{top}}(\mathcal{A}) \xrightarrow{\partial} K_0^{\mathrm{top}}\left(C_0^\infty((0, 1), \mathcal{A})\right) \to K_0^{\mathrm{top}}\left(C_0^\infty([0, 1), \mathcal{A})\right) = 0$$

□

Definition 11.19. *For every smooth algebra* $\mathcal{A}$ *and natural number* n*, define*

$$K_n^{\mathrm{top}}(\mathcal{A}) = K_0^{\mathrm{top}}\left((C_0^\infty(\mathbb{R}^n, \mathcal{A}))\right) \cong K_1^{\mathrm{top}}(\mathcal{A}).$$

Theorem 11.20. $K_n^{\mathrm{top}}(\mathcal{A}) \cong \pi_{n-1}(GL(\mathcal{A}))$ *for* $n \geq 1$.

Proof. We have

$$K_{\mathrm{n}}^{\mathrm{top}}(\mathcal{A}) = K_0^{\mathrm{top}}\left((C_0^\infty(\mathbb{R}^n,\mathcal{A}))^+\right) \cong K_1^{\mathrm{top}}\left(\left(C_0^\infty(\mathbb{R}^{n-1},\mathcal{A})\right)^+\right).$$

□

From here on out, assume that $\mathcal{A}$ is *commutative*. (Otherwise $SL\,(n,\mathcal{A})$ is not defined.)

Lemma 11.21. *(Milnor) Let A be an element of $M(n,\mathcal{A})$ with $n>1$ such that $\det(I+A)=1\in A$ and $\|a_{ij}\|<\frac{1}{n-1}$ for each matrix entry a_{ij} of A. Then $I+A$ can be expressed as a product of n^2+5n-6 elementary matrices, each of which depends continuously on A.*

Proposition 11.22. *$E(n,\mathcal{A})$ is the connected component of the identity of $SL(n,\mathcal{A})$.*

Proof. Every elementary matrix has determinant I, whence $E(n,\mathcal{A})$ is contained in $SL(n,\mathcal{A})$. Next, for each elementary matrix $e_{ij}(a)$, we have a path $\{e_{ij}(ta)\}$ from I to $e_{ij}(a)$, and thus $E(n,\mathcal{A})$ is contained in the connected component of the identity of $SL(n,\mathcal{A})$. The opposite containment follows from the Milnor lemma and the fact that any neighborhood of the identity generates all of the connected component of the identity in $SL\,(n,\mathcal{A})$. □

For each natural number n, let $\widetilde{E}_n$ denote the universal cover of $E(n,\mathcal{A})$ (assume it exists). Then we have a central extension

$$1 \longrightarrow \pi_1\left(E(n,\mathcal{A})\right) \longrightarrow \widetilde{E}_n \to E(n,\mathcal{A}) \longrightarrow 1$$

of groups. (Question for audience: why is $\pi_1\left(E(n,\mathcal{A})\right)$ abelian? Because $E(n,\mathcal{A})$ is a topological group.)

For each elementary matrix $e_{ij}(a)$ in $M(n,\mathcal{A})$, define $\tilde{e}_{ij}(a)$ in $\widetilde{E}_n$ to be the endpoint of the path that starts at I and covers the path $\{e_{ij}(ta)\}$. The elements $\tilde{e}_{ij}(a)$ satisfy the defining relations of $St(n,\mathcal{A})$, and thus there exists a group homorphism $\widetilde{\phi}_n : St(n,\mathcal{A}) \longrightarrow \widetilde{E}_n$. (That also comes from the fact that $St\,(n,\mathcal{A})$ is the universal central extension of $E(n,\mathcal{A})$.)

Proposition 11.23. *The homomorphism $\widetilde{\phi}_n$ is surjective and maps the kernel of the homomorphism $\phi_n : St(n,\mathcal{A}) \longrightarrow E(n,\mathcal{A})$ onto $\pi_1\left(E(n,\mathcal{A})\right)=\pi_1\left(SL(n,\mathcal{A})\right)=\pi_1\left(GL(n,\mathcal{A})\right)$.*

Proof. The previous proposition implies that $\pi_1\left(E(n,\mathcal{A})\right)=\pi_1\left(SL(n,\mathcal{A})\right)$, and the second equality follows from polar decomposition. To show that $\widetilde{\phi}$ is surjective, observe that the definition of $\tilde{e}_{ij}(A)$ implies that we can lift a neighborhood of the identity in $E(n,\mathcal{A})$ to an open subset of $\widetilde{E}_n$. This open subset generates $\widetilde{E}_n$ because $\widetilde{E}_n$ is connected. □

Corollary 11.24. *The limit homomorphism $\widetilde{\phi}$ of the $\widetilde{\phi}_n$ maps $K_2^{alg}(\mathcal{A})$ onto $K_2^{top}(\mathcal{A})$.*

(Recall $K_2^{top}\,(\mathcal{A}) := \pi_1\left(GL\,(\mathcal{A})\right)$.)

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129, USA