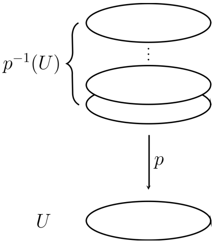
Polynomial Coverings

GAGA Seminar

Fall 2023

Definition

Let E and X be topological spaces and suppose $p : E \to X$ is a continuous surjection. An open subset U of X is *evenly covered* by p if $p^{-1}(U)$ is the disjoint union of a collection $\{V_\alpha\}_{\alpha \in A}$ of open subsets of E with the property that for each α in the index set A, the restriction of p to V_α is a homeomorphism from V_α to U.



Definition

If every point of X has a neighborhood that is evenly covered by p, then p is a *covering map* and that E is a *covering space* of X.

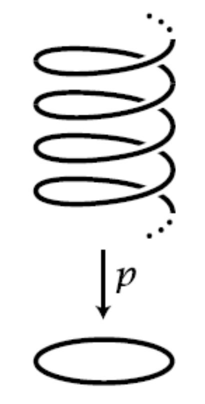
Example

For any nonempty topological space X and nonempty set A,

$$\pi : X \times A \to X,$$

where π is projection, is a covering map.

Example



$$p : \mathbb{R} \to S^1 \qquad p(x) = (\cos x, \sin x)$$

Definition

Suppose that $p : E \to X$ and $p' : E' \to X$ are covering maps. We say that E and E' are *equivalent* covering spaces if there exists a homeomorphism $h : E \to E'$ that makes the following diagram commute.

$$\begin{array}{ccc} E & \xrightarrow{\quad h \quad} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

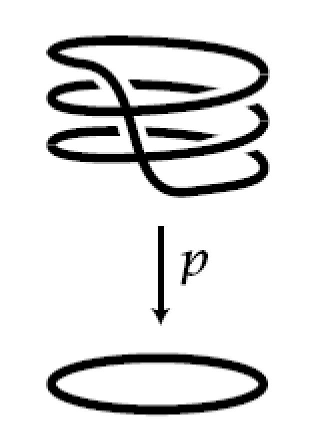
We say $p : E \to X$ is *trivial* if it is equivalent to our first example

$$\pi : X \times A \to X.$$

Definition

We say $p : E \to X$ is a *finite* cover if $p^{-1}(x)$ is a finite set for each x in X.

Example



Definition

Let X be a connected and locally path connected topological space, and let $C(X)$ be the ring of continuous complex-valued functions on X. A *Weierstrass polynomial* is a monic element of $C(X)[\lambda]$:

$$P(x, \lambda) = \lambda^n + \sum_{i=1}^{n} a_i(x)\lambda^{n-i}$$

If $P(x, \lambda)$ has distinct zeros for each x in X, we call $P(x, \lambda)$ a *simple* Weierstrass polynomial.

Example

For any X and any natural number n,

$$P(x, \lambda) = (\lambda - 1)(\lambda - 2) \cdots (\lambda - n)$$

is a simple Weierstrass polynomial of degree n over X.

Example

$$P(z, \lambda) = \lambda^n - z$$

is a simple Weierstrass polynomial of degree n over S^1.

Example

$$P((x, y, z), \lambda) = \lambda^2 - 2(x^2 + y^2 + iz^2)\lambda + 4i(x^2 + y^2)z^2$$

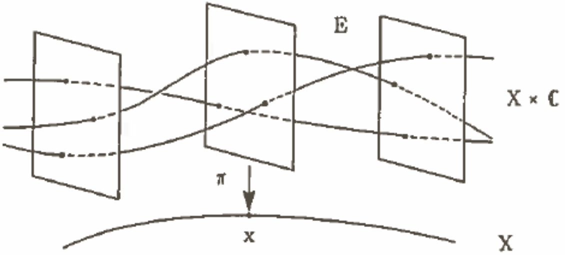
is a simple Weierstrass polynomial of degree 2 over S^2.

Definition

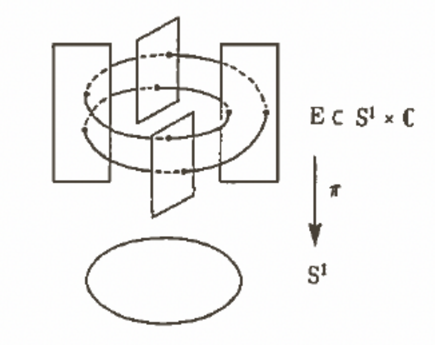
Let $P(x, \lambda)$ be a simple Weierstrass polynomial of degree $n > 1$ over X. Let

$$E = \{(x, \lambda) \in X \times \mathbb{C} : P(x, \lambda) = 0\},$$

and define $p : E \to X$ to be the restriction of the projection map $\pi : X \times \mathbb{C} \to X$. We call (E, p) the *n-fold polynomial covering space associated to* $P(x, \lambda)$.



$$P(z, \lambda) = \lambda^2 - z \in C(S^1)[\lambda]$$



Question

Is every finite covering space (equivalent to) a polynomial covering space?

Answer: No.

Question

Can we characterize which finite covering spaces are (equivalent to) a polynomial covering space?

Answer: Yes.

Definition

Let $n > 1$ be a natural number. The *discriminant set* of $\mathbb{C}^n$ is

$$\Delta = \{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n : P(\lambda) = \lambda^n + \sum_{i=1}^{n} a_i \lambda^{n-i} \text{ does not have distinct zeros}\}.$$

Definition

Let n be a natural number. The *elementary symmetric polynomials* in the variables $x_1, x_2, \ldots, x_n$ are defined to be

$$s_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$

$$s_2(x_1, x_2, \ldots, x_n) = \sum_{1 \le i < j \le n} x_i x_j$$

$$\vdots$$

$$s_n(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$$

Theorem

Every symmetric polynomial in n variables can be uniquely written as a polynomial in the elementary symmetric polynomials.

Example

$$x_1^2 + x_2^2 + \cdots + x_n^2 = (s_1(x_1, x_2, \ldots, x_n))^2 - 2s_2(x_1, x_2, \ldots, x_n)$$

Definition

Let δ be the unique polynomial in $n > 1$ variables satisfying the equation

$$\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 = \delta\big(-s_1(x_1, \ldots, x_n), s_2(x_1, \ldots, x_n), \ldots (-1)^n s_n(x_1, \ldots, x_n)\big).$$

The polynomial $\delta(a_1, a_2, \ldots, a_n)$ is called the *discriminant polynomial* in the variables $a_1, a_2, \ldots, a_n$. For a monic polynomial

$$P(\lambda) = \lambda^n + \sum_{i=1}^{n} a_i \lambda^{i-1}$$

in $\mathbb{C}[\lambda]$, the complex number $\delta(a_1, a_2, \ldots, a_n)$ is called the *discriminant* of $P(\lambda)$.

$$\delta(a_1, a_2) = a_1^2 - 4a_2$$

$$\delta(a_1, a_2, a_3) = a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 a_3 + 18a_1 a_2 a_3 - 27a_3^2$$

$$\begin{aligned}\delta(a_1, a_2, a_3, a_4) = 256a_4^3 &- 192a_1 a_3 a_4^2 - 128a_2^2 a_4^2 + 144a_2 a_3^2 a_4 - 27a_3^4 \\ &+ 144a_1^2 a_2 a_4^2 - 6a_1^2 a_3^2 a_4 - 80a_1 a_2^2 a_3 a_4 + 18a_1 a_2 a_2 a_4^3 + 16a_2^4 a_4 \\ &- 4a_2^3 a_3^2 - 27a_1^4 a_4^2 + 18a_1^3 a_2 a_3 a_4 - 4a_1^3 a_3^3 - 4a_1^2 a_2^3 a_4 + a_1^2 a_2^2 a_3^2\end{aligned}$$

Theorem

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be the roots of*

$$P(\lambda) = \lambda^n + \sum_{i=1}^{n} a_i \lambda^{n-i},$$

counted with multiplicity. Then for

$$a_i = (-1)^i s_i(\alpha_1, \alpha_2, \ldots, \alpha_n),$$

whence

$$\delta(a_1, a_2, \ldots a_n) = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2.$$

Corollary

$$(a_1, a_2, \ldots, a_n) \in \Delta \textit{ if and only if } \delta(a_1, a_2, \ldots a_n) = 0$$

Corollary

For each natural number $n > 1$, the discriminant set Δ is an algebraic variety of complex codimension one in $\mathbb{C}^n$.

Corollary

The complement B^n of the discriminant set in $\mathbb{C}^n$ is a connected and locally path-connected open subset of $\mathbb{C}^n$.

Definition

The *canonical Weierstrass polynomial* over B^n is

$$P^n\big((a_1, a_2, \ldots, a_n), \lambda\big) = \lambda^n + \sum_{i=1}^{n} a_i \lambda^{n-1},$$

and the associated n-fold polynomial covering map $p^n : E^n \to B^n$ is called the *canonical n-fold polynomial covering map*.

Definition

Let $p : E \to X$ be an n-fold polynomial covering map associated with the simple Weierstrass polynomial

$$P(x, \lambda) = \lambda^n + \sum_{i=1}^{n} a_i(x)\lambda^{n-i}$$

over X. The coefficient functions $a_i(x)$ of $P(x, \lambda)$ determine a map

$$a = (a_1, a_2, \ldots, a_n) : X \to B^n$$

called the *coefficient map* of $P(x, \lambda)$ and the polynomial covering map $p : E \to X$.

In this case, we will often write $P_a(x, \lambda)$ and $p_a : E_a \to X$.

Definition

$$a^*(E^n) = \Big\{\big(x, ((a_1, a_2, \ldots, a_n), \lambda)\big) \in X \times E^n : \\ p^n\big((a_1, a_2, \ldots, a_n), \lambda\big) = a(x)\Big\}$$

$$\begin{array}{ccc} a^*(E^n) & \xrightarrow{a^*} & E^n \\ {\scriptstyle (p^n)^*}\big\downarrow & & \big\downarrow{\scriptstyle p^n} \\ X & \xrightarrow{a} & B^n \end{array}$$

Theorem

Define $h : E_a \to a^*(E^n)$ *by the formula*

$$h(x, \lambda) = \big(x, ((a_1(x), a_2(x), \ldots, a_n(x)), \lambda)\big).$$

Then we have an equivalence of covering maps over X*:*

$$\begin{array}{ccc} E_a & \xrightarrow{\quad h \quad} & a^*(E^n) \\ & {\scriptstyle p_a} \searrow \quad \swarrow {\scriptstyle (p^n)^*} & \\ & X & \end{array}$$

Theorem (Pullback Criterion)

A n-fold covering map $p : E \to X$ *is equivalent to a polynomial covering map if and only if it is equivalent to the pullback of the canonical n-fold covering map* $p^n : E^n \to B^n$.

Definition

We say that a finite covering map $p : E \to X$ can be imbedded in the trivial complex line bundle $X \times \mathbb{C}$ over X if there exists an imbedding $f : E \to X \times \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & X \times \mathbb{C} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

commutes.

Without loss of generality, we may assume that E is a subset of $X \times \mathbb{C}$ and that p is the restriction of the projection π to E.

Theorem (Imbedding Criterion)

A finite covering map $p : E \to X$ is equivalent to a polynomial covering map if and only if it admits an imbedding into the trivial line bundle over X.

Proof.

($\Longrightarrow$) By its very definition, a polynomial covering map imbeds into $X \times \mathbb{C}$.

($\Longleftarrow$) Let $p : E \to X$ be an n-fold covering map that imbeds into $X \times \mathbb{C}$, and view E as a subset of $X \times \mathbb{C}$. Define

$$P(x, \lambda) = \prod_{(x,\lambda_x) \in p^{-1}(x)} (\lambda - \lambda_x).$$

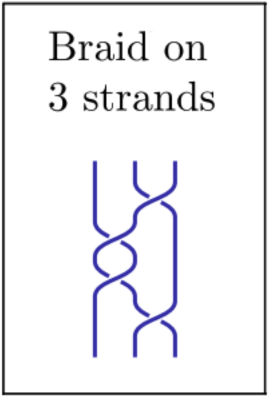
Then $P(x, \lambda)$ is a simple Weierstrass polynomial, and the local triviality of E implies that the coefficient functions $a_i : X \to \mathbb{C}$ are continuous. The covering map $p : E \to X$ is the polynomial covering map associated to $P(x, \lambda)$. □

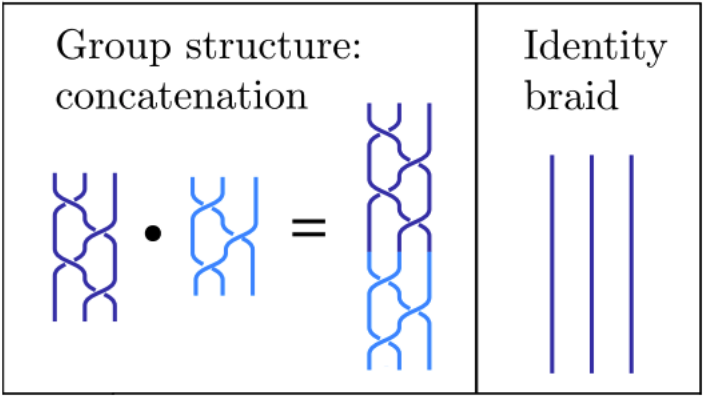
Example

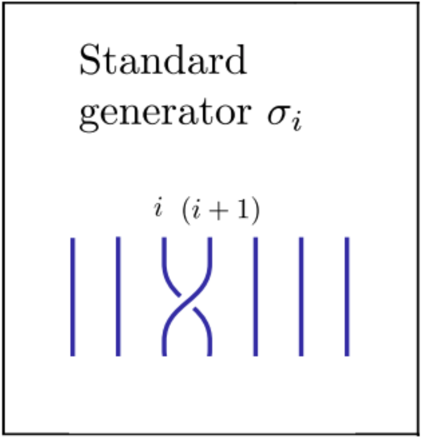
$$p : S^1 \to \mathbb{R}P^1$$

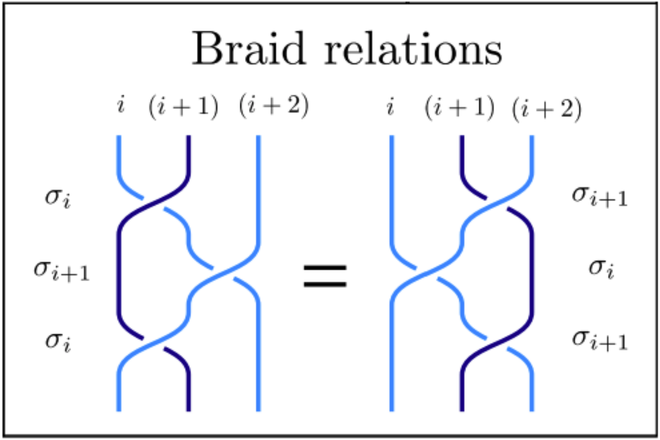
This is the nontrivial double cover of the circle, which is equivalent to the polynomial cover associated to $\lambda^2 - z$.

To construct nonexamples, we need to talk about *braids*.


Braid on
3 strands


Group structure:
concatenation
Identity
braid


Standard
generator σ_i
i $(i+1)$


Braid relations
i
(i + 1)
(i + 2)
i
(i + 1)
(i + 2)
σi
σi+1
σi
σi+1
σi
σi+1
=

Definition

Let $n \geq 2$. The *braid group on n strands* is denoted by B_n and is the group with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for all } i$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for all } |i - j| > 1$$

Theorem

B_n is torsion-free.

Idea of Proof.

The map that sends each generator σ_i to 1 defines a homomorphism from B_n to $\mathbb{Z}$, and $\mathbb{Z}$ is torsion-free. □

Theorem

$\pi_1(B^n) \cong B(n)$

Theorem

Suppose that $\pi_1(X)$ is a torsion group; i.e., every group element has finite order. Then every polynomial covering of X is trivial.

Corollary

For $n \geq 2$, the quotient map $\pi : S^n \to \mathbb{R}P^n$ is not equivalent to a polynomial covering map.

On the other hand,

Theorem

Suppose that $\pi_1(X)$ is a free group. Then every finite covering map is equivalent to a polynomial covering map.

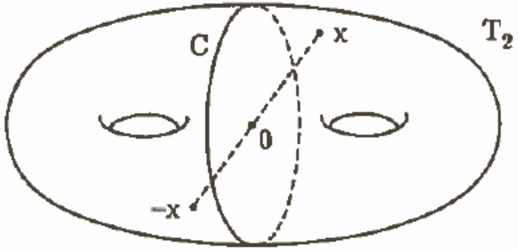
Corollary

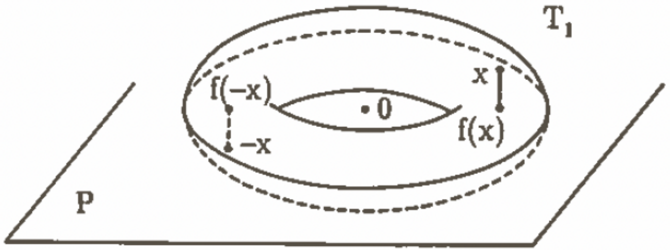
Every finite cover of S^1 is equivalent to a polynomial covering map.

One more example for the geometric topologists!

Lemma

Let T_g be a closed orientable surface of genus g, viewed as a subset of $\mathbb{R}^3$ that is symmetric with respect to the origin. There exists a continuous map $f : T_g \to \mathbb{C}$ with the property that $f(x) \neq f(-x)$ for all x in T_g if and only if g is odd.


C
x
T_2
0
-x


T1
x
f(−x)
• 0
f(x)
−x
P

Theorem

Let U_g be a closed nonorientable surface of genus g. The orientation double cover $\pi : T_g \to U_{g+1}$ is equivalent to a polynomial cover if and only if g is odd.

Proof.

Suppose the orientation double cover $\pi : T_g \to U_g$ is equivalent to a polynomial cover. Then there exists an imbedding $h = (\pi, f) : T_g \to U_{g+1} \times \mathbb{C}$, and $f : T_g \to \mathbb{C}$ has the feature that $f(x) \neq f(-x)$ for all x in T_g. Conversely, if such an f exists, $h = (\pi, f)$ is an imbedding of T_g into $U_{g+1} \times \mathbb{C}$. □