NOVIKOV ADDITIVITY

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1. Novikov Additivity and Wall Non-additivity

Given two manifolds M_1, M_2 glued together along a common boundary. Additivity holds when the signature is additive with respect to this decomposition. Nonadditivity occurs when a manifold with boundary M is partitioned into two manifolds M_1 and M_2 with corners, and there is a formula

$$\sigma(M) = \sigma(M_1) + \sigma(M_2) + \text{Maslov},$$

where Maslov is a Maslov index. We now proceed.

1.1. **Bilinear forms.**

On finite dimensional $\mathbb{R}$-vector spaces, given a bilinear form

$$\phi : V \otimes V \to \mathbb{R},$$

we call it symmetric if $\phi(v,w) = \phi(w,v)$ for all $v, w \in V$. The matrix representation is

$$M_{ij} = \phi(e_i, e_j).$$

Let

$$\begin{aligned} \sigma(V,\phi) &= \sigma(V) = \dim(\text{largest pos. def. subspace}) - \dim(\text{largest neg. def. subspace}) \\ &= \#(\text{pos. eigenvalues}) - \#(\text{neg. eigenvalues}). \end{aligned}$$

We say that ϕ is nondegenerate if $\phi(v,w) = 0$ for all w implies $v = 0$. We say ϕ is nonsingular (same) iff

$$\begin{aligned} V &\cong \operatorname{Hom}(V,\mathbb{R}) \\ v &\mapsto \phi(v,\cdot). \end{aligned}$$

Fun facts:

- (V_1,ϕ_1), (V_2,ϕ_2) produces $\phi_1 \boxplus \phi_2$ on $V_1 \oplus V_2$: $\begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}$. The signature of the sum is the sum of the signatures.
- On $V_1 \otimes V_2$, there is a natural form. The signature $\sigma(\phi_1 \otimes \phi_2) = \sigma(\phi_1)\,\sigma(\phi_2)$.
- Suppose ϕ is nondegenerate. Then $\sigma(\phi) = 0$ iff there exists a self-annihilating subspace $A \subset V$ such that $\dim(A) = \frac{1}{2}\dim(V)$. Self-annihilating means $A = A^{\perp}$, i.e. $\phi(a,b) = 0$ for all $a, b \in A$.

Topological Connections

Let M be a closed, connected, oriented, $4n$-manifold. Then there is a bilinear form on $H^{2n}(M) \otimes H^{2n}(M) \xrightarrow{\cup} \mathbb{R}$. The cup product is symmetric and nondegenerate and implements Poincaré duality. Equivalently,

$$H_{2n}(M) \otimes H_{2n}(M) \xrightarrow{\pitchfork} \mathbb{R}$$

is the intersection pairing. If M is smooth, you can represent chains by chains that intersect nicely.

Given M, define the signature of the manifold to be

$$\sigma(M) = \sigma(\pitchfork) = \sigma(\cup).$$

If $\dim M \neq 0 \bmod 4$ then $\sigma(M) = 0$.

Fun Facts:

- Reversing orientation: $\sigma(-M) = -\sigma(M)$.
- $\sigma(M \times N) = \sigma(M)\sigma(N)$
- If $M^{4n} = \partial N^{4n+1}$, then $\sigma(M) = 0$. This comes from the exact sequence

$$H_{2n}(M) \to H_{2n}(N) \to H_{2n}(N, M) \to H_{2n-1}(M).$$

Signature of manifolds with boundary

Let M^{4n} be a compact, connected oriented manifold with boundary. Then there is a map

$$H_{2n}(M) \to H_{2n}(M, \partial M) \cong \mathrm{Hom}(H_{2n}(M), \mathbb{R})$$

by Lefschetz duality. This is not necessarily an isomorphism, so $\pitchfork$ is not necessarily nondegenerate anymore. To fix this, the claim is that $\pitchfork$ is nondegenerate on

$$V \diagup W = H_{2n}(M) \diagup \mathrm{Im}(H_{2n}(\partial M) \to H_{2n}(M)) \cong \mathrm{Im}(H_{2n}(M) \to H_{2n}(M, \partial M)).$$

To see this, suppose that $v \in V$, $w \in W$, $v \pitchfork w = 0$ by pushing the boundary and interior away from each other. So

$$v + W \pitchfork v' + W = v \pitchfork v' + W$$

is a well-defined pairing. To see nondegeneracy, suppose that $v \in V \diagup W$ and $v' \in V \diagup W$. If $v \pitchfork v' = 0 \mathrm{mod} W$ for all v', then $v \pitchfork v' = i(v) \pitchfork v'$ with i the "push-in map". But $i(v) \in H_{2n}(M, \partial M) \cong H_2(M)^*$, so that $i(v) = 0$. But then $v \in \ker(H_{2n}(M) \to H_{2n}(M, \partial M))$, so $v \in \mathrm{Im}(H_{2n}(\partial M) \to H_{2n}(M))$, so $v \in W$.

Proposition 1.1. $\sigma(\partial M^{4n+1}) = 0$.

Proof. This follows from the fact that if Φ is a nondegenerate bilinear symmetric form and $A \subset V$ with $\Phi(A, A) = 0$ and $\dim A = \frac{1}{2}\dim V$ iff $A = A^{\perp}$.

The key observation is that if x^{2n} and y^{2n} are two chains in general position on the boundary, and we wish to compute $x \pitchfork_{\partial M} y$. Suppose in addition that $y = \partial Y$. Then this is the same as $x \pitchfork_M Y$. Let $K = \ker(H_{2n}(\partial M) \to H_{2n}(M))$, which are the cycles in ∂M that bound in M. Claim: $K = K^{\perp}$. Suppose $x, y \in K$. Then $x \pitchfork_{\partial M} y = x \pitchfork_M Y$. Since $\pitchfork_M \colon H_{2n}(M) \otimes H_{2n+1}(M, \partial M) \to \mathbb{R}$ is well-defined, $x \pitchfork_M Y = 0$. So $K \subset K^{\perp}$. Suppose that $x \notin K$. We will show that $x \notin K^{\perp}$. Since $x \notin K$, x is a nonzero element of $H_{2n}(M)$. By Poincaré duality, there exists $Y \in H_{2n+1}(M, \partial M)$ such that $x \pitchfork_M Y = x \pitchfork_{\partial M} y \neq 0$. So $y \in K$, and $x \pitchfork_{\partial M} y \neq 0$. □

2. Discussion of Novikov Additivity

Let $M = M_1 \cup_{\partial M_1 = \partial M_2} M_2$. The claim is $\sigma(M) = \sigma(M_1) + \sigma(M_2)$. Here $\sigma(M_j)$ is the signature of the of the $\pitchfork$ form on

$$H_{2n}(M_j) \diagup \mathrm{Im}(H_{2n}(\partial M_j) \to H_{2n}(M_j)) \cong \mathrm{Im}(H_{2n}(M_j) \to H_{2n}(M_j, \partial M_j)).$$

The rough idea is as follows. There are several different kinds of chains on M, depending how they interest the boundary. Let A_i be the image $A_i = \mathrm{Im}((H_{2n}(M_i)) \to H(M))$. Then $A_1 \cap A_2 = \mathrm{Im}((H(\partial M)) \to H(M))$. Note that $A_1 \pitchfork A_2 = 0$. We have

$$A_1 \cap A_2 = (A_1 + A_2)^{\perp}.$$

If you buy this, $H_{2n}(M)\diagup(A_1+A_2)\cong(A_1\cap A_2)^*$. Then

$$\begin{aligned}(A_1+A_2)\diagup(A_1\cap A_2) &= A_1\diagup(A_1\cap A_2)\oplus A_2\diagup(A_1\cap A_2)\\ &\cong \operatorname{Im}(H(M_1)\to H(M_1,\partial M_1))\oplus\operatorname{Im}(H(M_2)\to H(M_2,\partial M_2))\\ &= I_1\oplus I_2\end{aligned}$$

Then $A_1\cap A_2\subset A_1+A_2\subset H(M)$. Then

$$\begin{aligned}H(M) &= A_1\cap A_2\oplus(A_1+A_2)\diagup(A_1\cap A_2)\oplus H(M)\diagup(A_1+A_2)\\ &= A_1\cap A_2\oplus I_1+I_2\oplus(A_1\cap A_2)^*\oplus(A_1\cap A_2).\end{aligned}$$

The intersection form acts on this decomposition as

$$\pitchfork_M=\begin{pmatrix}\pitchfork_{M_1} & 0 & * & 0\\ 0 & \pitchfork_{M_2} & * & 0\\ * & * & * & *\\ 0 & 0 & * & 0\end{pmatrix}$$
$$\leftrightarrow\begin{pmatrix}\pitchfork_{M_1} & 0 & 0 & 0\\ 0 & \pitchfork_{M_2} & 0 & 0\\ 0 & 0 & * & *\\ 0 & 0 & * & 0\end{pmatrix}$$

(similarity). But then

$$\sigma(M)=\sigma(M_1)+\sigma(M_2)+0$$

(The last part is zero because of the existence of a self-annihilating subspace, $\sigma(\partial N)=0$.)

Also, Novikov additivity holds for cylinders.

The harder case is where there is a manifold with boundary M, and the boundary is cut as well.

$$M=M_1\cup M_2.$$

This is Wall non-additivity.

$$\sigma(M)=\sigma(M_1)+\sigma(M_2)+\sigma(V;A,B,C),$$

where the last term is a Maslov index. Here V is a symplectic vector space, and A is a Lagrangian subspace (or at least isotropic). This comes with the intersection of ∂M_j with ∂M.

3. Maslov Indices and Wall nonadditivity

Novikov additivity: If $M=M_1\cup M_2$, $\sigma(M)=\sigma(M_1)+\sigma(M_2)$ if M has no boundary.

Wall nonadditivity: If Y^{4n} has boundary, $X_0=\partial Y_\pm$, $X_\pm=X\cap\partial Y\cap Y_\pm$, $Z=\partial X_\pm$

$$\sigma(Y)=\sigma(Y_+)+\sigma(Y_-)+\sigma(V;A,B,C).$$

The Maslov triple index correction is $\sigma(V;A,B,C)$. In general, V is a vector space with an antisymmetric pairing Φ, and A, B, C are self-annihilating subspaces of V. For Wall,

$$\begin{aligned}V &= H_{2n-1}(Z)\\ A &= \ker(V\to H_{2n-1}(X_-))\\ B &= \ker(V\to H_{2n-1}(X_+))\\ C &= \ker(V\to H_{2n-1}(X_0))\end{aligned}$$

The Maslov index is defined as follows. Let

$$W=\frac{A\cap(B+C)}{A\cap B+A\cap C}.$$

This is symmetric (up to isomorphism) in A,B,C. An element in W is represented by a triple (a,b,c) such that $a+b+c=0$. We construct an isomorphism

$$W\to\frac{B\cap(A+C)}{B\cap A+B\cap C}.$$

Let $f(a)=b$ where $a+b+c=0$. Suppose that on the other hand, $a+b+c=0$, $a+b'+c'=0$. Then $b-b'=c-c'\in B\cap C$, so the quotient kills the ambiguity. So the map is well-defined. The kernel of this map $A\cap(B+C)\to\frac{B\cap(A+C)}{B\cap A+B\cap C}$. Then $a+c=0$, so $a\in A\cap C$, so there is no kernel. Also, it is clearly onto. Also $A\cap B$ are the same in the two pieces, so the map is an isomorphism.

The pairing on W is defined as follows. Given $a+b+c=0$, $a'+b'+c'=0$, we have

$$\begin{aligned}0&=\Phi(0,a')=\Phi(a+b+c,a')=\Phi(b+c,a'),\\ \Phi(b,a')&=-\Phi(c,a')\\ &=\Phi(c,b')\\ &=\Phi(a,b')=\Phi(a,c')=\Phi(b,c').\end{aligned}$$

We define Ψ' on $A\cap(B+C)$ by

$$\Psi'(a,a')=\Phi(a,b').$$

It turns out this is well-defined in b', because if $a'+b''+c''=0$,

$$\begin{aligned}\Phi(a,b')-\Phi(a.b'')&=\Phi(a,b'-b'')\\ &=-\Phi(c,a'-a')=0.\end{aligned}$$

A similar argument shows that it is well-defined in the first variable. Now, Ψ' descends to a well-defined Ψ on W. We see that if $a'\in A\cap C$, then $a'+c'=0$, so $b'=0$, so $\Psi'(a')=0$. The same argument works for $A\cap B$, using the appropriate symmetry. We now show Ψ is symmetric on W:

$$\begin{aligned}\Psi(a,a')-\Psi(a',a)&=\Phi(a,b')-\Phi(a',b)\\ &=\Phi(a,b')-\Phi(b,a')\\ &=\Phi(a+b,a'+b')-\Phi(a,a')-\Phi(b,b')\\ &=\Phi(-c,-c')=0.\end{aligned}$$

Now, we define Ψ as a symmetric pairing on W, and we define

$$\sigma(V_\Phi;A,B,C):=\sigma(\Psi).$$

Back to topology: we can compute the signature of the pieces by looking at

$$L=\operatorname{Im}(H_{2n}(X)\to H_{2n}(Y,\partial Y))\big/\text{radical}.$$

Every $x\in L$ can be represented by a chain x_2 in X_0 that has boundary in Z. We get a map $L\to W$. We take

$$x_2\to\partial x_2\in H_{2n-1}(Z)=V\twoheadrightarrow W$$

which works, since $\partial x_2 \in B \cap (A + C)$ (check: in B by defn, it suffices then to show that $x_2 \mapsto 0 \in H(X_+ \cup X_-)$, and

$$H(X) \to H_{2n}(Y) \xrightarrow{\partial} H_{2n}(Y, \partial Y) \to H(X_+ \cup X_-)$$

.) In the end, $L \cong W$. We need to show that $(L, \pitchfork) \cong (W, \Psi)$, then $\sigma(L) = \sigma(V; A, B, C)$.

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