THE DICTIONARY BETWEEN COMMUTATIVE AND NONCOMMUTATIVE GEOMETRY AND TOPOLOGY

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1. C^* algebras

Let X be a compact Hausdorff space. Let $C(X)$ be a $\mathbb{C}$-algebra of continuous $\mathbb{C}$-valued functions on X. We have the involution $f^*(x) = \overline{f(x)}$ and the norm $\|f\| = sup\{|f(x)| : x \in X\}$. Also, $C(X)$ is a Banach $*$-algebra (normed complete $*$-algebra).

There is a contravariant functor C from the category of compact, Hausdorff spaces and continuous functions to Banach $*$-algebras and $*$-homomorphisms. A C^*-algebra A is a Banach $*$-algebra where $\|a^*a\| = \|a\|^2$ for all $a \in A$. The space $C(X)$ is a commutative C^*-algebra.

Theorem 1.1. *(Gelfand-Naimark) Every commutative C^*-algebra with unit is $*$-isomorphic to $C(X)$ for some compact Hausdorff space X.*

Theorem 1.2. *Every closed $*$-ideal of $C(X)$ uniquely has the form*

$$C_0(X \diagdown A) := \{f \in C(X) : f(a) = 0 \text{ for all } a \in A\}$$

for some unique closed subset A.

Corollary 1.3. *The maximal ideals in $C(X)$ can be identified with points of X.*

Corollary 1.4. *The space X can be recovered from $C(X)$.*

Theorem 1.5. *The functor C determines a category equivalence between compact Hausdorff spaces and commutative C^*-algebras.*

In theory, we could do topology by working with C^*-algebras, but in practice this usually does not work well. One good example is as follows. $\{\mathbb{C}$-vector bundles over $X\}$ corresponds to $\{$finitely generated projective modules over $C(X).\}$ (Serre-Swan Theorem).

"Non-commutative topology" can be viewed as the study of general unital C^*-algebras — ie noncommutative ones. Why can't you learn more topology from the noncommutative side? There are many maps between topological spaces. However, the C^* condition is very strong, and there is a lot of rigidity: not many $*$-homomorphisms in the noncommutative case.

A more modern idea (Connes): study "bad" topological spaces(i.e. nonHausdorff), by replacing them with "good", but noncommutative, C^*-algebras.

Examples:

(1) Orbit space of a (not necessarily compact) Lie group acting on a compact manifold.
(2) Leaf space of a foliation.
(3) Space of irreducible representations of a discrete or Lie group on a Hilbert space.

The C^*-algebra for a group G acting on a compact manifold M via $\alpha : G \to \mathrm{Aut}(M)$ is

$$C_c(G, C(M)) = \{\text{continous fcns } \phi : G \to C(M) \text{ with compact support}\}$$

with convolution product

$$(\phi * \psi)(g) = \int \phi(h)\, \alpha_h\left(\psi\left(h^{-1}g\right)\right) dh$$

with pointwise addition. We complete $C_c(G, C(M))$ to a C^*-algebra.

If Γ is a discrete group, $\mathbb{C}\Gamma$ acts on $\ell^2\Gamma$, i.e. $\mathbb{C}\Gamma \subseteq \ell(\ell^2\Gamma)$. Then

$$C_r^*(\Gamma) = \text{closure of } \mathbb{C}\Gamma \text{ in } \ell\left(\ell^2\Gamma\right)$$

is known as the **reduced group C^*-algebra**. The simplest case is $C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$ via the Fourier series.

2. Cyclic Homology

Let A be a $\mathbb{C}$-algebra with unit. Let

$$C_n^\lambda(A) = \bigotimes_{n+1} A \diagup \sim,$$

where

$$a_n \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \sim (-1)^n a_0 \otimes a_1 \otimes \ldots \otimes a_n.$$

We have the boundary map

$$\begin{aligned} b &: C_n^\lambda(A) \to C_{n-1}^\lambda(A) \\ b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_0 a_1 \otimes \ldots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ &\quad + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_n. \end{aligned}$$

Then $b^2 = 0$, and the cyclic homology of A is defined to be $H_*^\lambda(A) = \text{homology of } \left(C_*^\lambda(A), b\right)$.

We also write elements $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ as noncommutative differential forms

$$a_0 da_1 da_2 \ldots da_n.$$

When $A = C^\infty(M)$, this produces isomorphisms

$$\begin{aligned} H_{2n}^\lambda(C^\infty(M)) &\cong H_{dR}^{\text{even}}(M; \mathbb{C}) \\ H_{2n+1}^\lambda(C^\infty(M)) &\cong H_{dR}^{\text{odd}}(M; \mathbb{C}) \end{aligned}$$

for n sufficiently large. So cyclic homology is a way of making sense of differential forms when you don't have a smooth manifold. More precisely,

$$H_k^\lambda(C^\infty(M)) = \Omega^k(M) \diagup d\left(\Omega^{k-1}(M)\right) \oplus H_{dR}^{k-2}(M) \oplus H_{dR}^{k-4}(M) \oplus \ldots$$

Other ways of getting cyclic homology are as follows. Question: where do elements of $H_*^\lambda(A)$ come from. Answer: K-theory. Let e (determines class in $K_0(A)$) be an idempotent in $M(m, A)$, then

$$Tr\left(e\,(de)^n\right) \in H_n^\lambda(A)$$

for n even. Let u (determines class in $K_1(A)$) be an element of $GL(m, A)$. Then

$$Tr\left(\left(u^{-1}du\right)^n\right) \in H_n^\lambda(A)$$

for n odd. Think of e as a projection from a trivial bundle to a vector bundle.

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)\left(\begin{array}{cc} da_{11} & da_{12} \\ da_{21} & da_{22} \end{array}\right)^n .$$

Next, consider cyclic cohomology. Let A be a topological algebra with unit. Let $C_\lambda^n(A)$ be the A-module of continuous multilinear maps $A^{n+1} \to \mathbb{C}$. Let

$$\begin{aligned} b &: C_\lambda^n(A) \to C_\lambda^{n+1}(A) \\ (b\phi)(a_0 \otimes a_1 \otimes \ldots \otimes a_{n+1}) &= \phi(a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) \\ &\quad + \sum_{i+1}^{n-1} (-1)^i \phi(a_0 \otimes a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) \\ &\quad + (-1)^n \phi(a_n a_0 \otimes a_1 \otimes \ldots \otimes a_n). \end{aligned}$$

Then $b^2 = 0$, and cyclic cohomology is defined to be

$$H_\lambda^*(A) = \text{cohomology of } \left(C_\lambda^n(A), b\right).$$

There is a pairing

$$H_\lambda^*(A) \times H_*^\lambda(A) \to \mathbb{C}.$$

Question: where do interesting elements of cyclic cohomology come from? Answer: From Fredholm modules. A **Fredholm module** over A is a triple $(\mathcal{H}, \pi, F)$, where $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space with grading operator ε ($\varepsilon^2 = 1$,$\mathcal{H}_+ =$ 1-eigenspace, $\mathcal{H}_- = (-1)$-eigenspace; $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation of A on $\mathcal{H}$ that respects the grading:

$$\pi(a) = \left(\begin{array}{cc} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{array}\right);$$

$F \in \mathcal{B}(\mathcal{H})$, $F^2 - 1 \in \mathcal{K}(\mathcal{H})$, $F\pi(a) - \pi(a) F \in \mathcal{K}(\mathcal{H})$ for all $a \in A$, $\varepsilon F = -F\varepsilon$,

$$F = \left(\begin{array}{cc} 0 & * \\ * & 0 \end{array}\right).$$

If $F\pi(a) - \pi(a) F \in \mathcal{L}^p(\mathcal{H})$ (ie p^{th} power is trace class, $p \geq 1$) for all $a \in A$, we say $(\mathcal{H}, \pi, F)$ is p-summable. If $F\pi(a) - \pi(a) F \in \mathcal{L}^p(\mathcal{H})$ for all $a \in \mathcal{A} \subseteq A$ for a dense subset, we say $(\mathcal{H}, \pi, F)$ is essentially p-summable.

Prototypical example: $A = C(M)$, M smooth compact manifold, $\mathcal{H} = L^2(M, E)$, with E a $\mathbb{Z}_2$-graded Hermitian vector bundle over M, and A acts on $\mathcal{H}$ by pointwise multiplication. Then D is an elliptic (pseudo)differential operator on E of the form $\left(\begin{array}{cc} 0 & * \\ * & 0 \end{array}\right)$. On $\mathbb{T}^2$,

$$D = \left(\begin{array}{cc} 0 & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \end{array}\right)$$

is an example. Let $F =$ positive spectral projection of D if D is essentially self-adjoint, or

$$F = D\left(1 + D^2\right)^{-1/2},$$

so that $F^2 - I \in \mathcal{K}$. For example,

$$F\left(\sum_{nse\mathbb{Z}} a_n e^{in\theta}\right) = \sum_{n \geq 0} a_n e^{in\theta}.$$

Note that $(\mathcal{H}, \pi, F)$ is essentially p-summable for $p > \dim M$.

3. Answer to Igor's Question

Let $A = C^{\infty}(M)$, M a smooth compact manifold. Consider the double complex:

$$\begin{array}{cccccc} & \downarrow^{b} & & \downarrow^{b} & & \downarrow^{b} \\ \overset{B}{\longleftarrow} & A\otimes A\otimes A & \overset{B}{\longleftarrow} & A\otimes A & \overset{B}{\longleftarrow} & A \\ & \downarrow^{b} & & \downarrow^{b} & & \\ & A\otimes A & \overset{B}{\longleftarrow} & A & & \\ & \downarrow^{b} & & & & \\ & A & & & & \end{array}$$

Let

$$B\left(a_0\otimes a_1\otimes ...\otimes a_n\right) = \sum_{i=0}^{n-1}\left[\begin{array}{c} (-1)^{ni}\left(1\otimes a_i\otimes ...\otimes a_n\otimes a_0\otimes ...\otimes a_{i-1}\right) \\ -(-1)^{n(i-1)}\left(a_{i-1}\otimes 1\otimes a_i\otimes ...\otimes a_{i-2}\right)\end{array}\right]$$

Then $B^2 = 0$, $Bb + bB = 0$, where

$$\begin{aligned} b\left(a_0\otimes a_1\otimes ...\otimes a_n\right) &= a_0a_1\otimes ...\otimes a_n \\ &\quad +\sum_{i=1}^{n-1}(-1)^i a_0\otimes a_1\otimes ...\otimes a_{i-1}\otimes a_ia_{i+1}\otimes ...\otimes a_n \\ &\quad +(-1)^n a_na_0\otimes a_1\otimes ...\otimes a_n. \end{aligned}$$

This complex is called $\mathcal{B}(A)$, and Tot$(\mathcal{B}(A))$ is the complex obtained by taking direct sums on the diagonal. You can do the same thing with the Cech-de Rham complex.

Theorem 3.1. $H_*\left(\mathrm{Tot}\left(\mathcal{B}(A)\right)\right)\cong H_*^{\lambda}(A)$.

The truncated de Rham complex is

$$\begin{array}{cccccc} & \downarrow^{0} & & \downarrow^{0} & & \downarrow^{0} \\ \overset{d}{\longleftarrow} & \Omega^2(M) & \overset{d}{\longleftarrow} & \Omega^1(M) & \overset{d}{\longleftarrow} & \Omega^0(M) \\ & \downarrow^{0} & & \downarrow^{0} & & \\ & \Omega^1(M) & \overset{d}{\longleftarrow} & \Omega^0(M) & & \\ & \downarrow^{0} & & & & \\ & \Omega^0(M) & & & & \end{array}$$

One can check that $d^2 = 0$, $0^2 = 0$, $0d + d0 = 0$. Call this complex $\mathcal{D}(M)$.

Theorem 3.2. $H^*\left(\mathrm{Tot}\left(\mathcal{D}(M)\right)\right)\cong H_{dR}^*(M)$.

Define $\pi_n : \bigotimes_{n+1} A \to \Omega^n(M)$ by

$$\pi_n\left(a_0\otimes ...\otimes a_n\right) = a_0da_1...da_n\ .$$

Then $\left\{\frac{1}{n!}\pi_n\right\}$ determines a map from $\mathcal{B}(A)$ to $\mathcal{D}(M)$ that induces an isomorphism.

4. More Fun with Fredholm Modules

Recall: a Fredholm module over a unital $\mathbb{C}$-algebra A is a triple $(\mathcal{H},\pi,F)$, where

- $\mathcal{H} = \mathcal{H}^+\oplus\mathcal{H}^-$ is a $\mathbb{Z}_2$-graded Hilbert space with grading operator ε ($\varepsilon^2 = 1$).

- $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation of A on $\mathcal{H}$ and respects the grading, i.e.

$$\pi(a) = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix}$$

- $F \in \mathcal{B}(\mathcal{H})$, $F^2 - I$ is compact, F reverses the grading, and $[F, \pi(a)]$ is compact for each $a \in A$.

$$F = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}.$$

Note that compact means a (operator norm) limit of finite rank operators.

(Think: $F = D(1 + D^2)^{-1/2}$, D Dirac operator, $A = C^\infty(M)$). If $[F, \pi(a)]$ is trace class, we say $(\mathcal{H}, \pi, F)$ is **1-summable**. If this condition only holds for a dense subalgebra $\mathcal{A}$ of A, we say that this module is **essentially** 1-summable.

The **character** of an essentially 1-summable Fredholm module over A is

$$\rho(a) = \frac{1}{2} Trace(\varepsilon F [F, \pi(a)]).$$

This ρ determines an element of $H^1_\lambda(A)$. Important commutative diagram:

$$\begin{array}{ccccc} Fred(A) & \times & K_*(A) & \overset{\text{index}}{\longrightarrow} & \mathbb{Z} \\ \downarrow^{\text{ch}} & & \downarrow^{\text{ch}} & & \downarrow \\ H^*_\lambda(A) & \times & H^\lambda_*(A) & \to & \mathbb{C} \end{array}$$

Picking a Fredholm module is akin to choosing a Riemannian structure.

Application: Let Γ be a discrete group, and let $\mathbb{C}\Gamma$ be the complex group algebra. Let $\mathbb{C}\Gamma \subseteq \mathcal{B}(\ell^2(\Gamma))$ be the left regular representation. Then

$$\mathbb{C}\Gamma = \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma : a_\gamma \in \mathbb{C} \right\}.$$

Then

$$a_\gamma \gamma : \ell^2(\Gamma) \to \ell^2(\Gamma)$$

is defined by

$$a_\gamma \gamma(\delta_\alpha) = a_\gamma \delta_{\gamma\alpha}$$

The norm closure of $\mathbb{C}\Gamma$ in $\mathcal{B}(\ell^2(\Gamma))$ is called the reduced C*-algebra $C^*_r(\Gamma)$ of Γ.

Noncommutative connectivity conjecture:

Conjecture 4.1. *(Bass Idempotent Conjecture): If Γ is torsion-free, then $\mathbb{C}\Gamma$ has no nontrivial idempotents (i.e. $e \neq 0, 1$).*

Conjecture 4.2. *(Kadison Conjecture): If Γ is torsion-free, then $C^*_r\Gamma$ has no nontrivial idempotents.*

(Note Baum-Connes Conjecture implies both of these and the Borel Conjecture and ...)

5. A proof of Kadison's conjecture for F_2

Let F_2 be the free group on two generators.

Let $\mathbb{C}F_2 \subseteq C_r^*(F_2) \subset \mathcal{B}(\ell^2(F_2))$ - reduced group C^*-algebra.

Here,

$$\sum_{\gamma \in F_2} a_\gamma \gamma \quad : \quad \ell^2(F_2) \to \ell^2(F_2)$$

$$\left(\sum_{\gamma \in F_2} a_\gamma \gamma\right)(\delta_\alpha) \;=\; \sum_{\gamma \in F_2} a_\gamma \gamma\alpha,$$

where

$$\langle \delta_\alpha, \delta_\beta \rangle = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

Theorem 5.1. *(Kadison Conjecture): $C_r^*(F_2)$ contains no nontrivial idempotents.*

Definition 5.2. *Let $\tau : A \to \mathbb{C}$ be a trace on a C^*-algebra A ($\tau(ab) = \tau(ba)$). We say τ is*

- ***positive*** *if $\tau(a^*a) \geq 0$ for all $a \in A$.*
- ***faithful*** *if $\tau(a^*a) = 0$ iff $a = 0$.*

Example 5.3. *The function $\tau : \mathbb{C}F_2 \to \mathbb{C}$ defined by*

$$\tau\left(\sum_{\gamma \in F_2} a_\gamma \gamma\right) = a_1$$

extends to a positive faithful trace on $C_r^*(F_2)$.

Theorem 5.4. *Let A be a C^*-algebra that admits a positive faithful trace τ such that $\tau(1) = 1$. Let $(\mathcal{H}, \pi, F)$ be an essentially 1-summable Fredholm module on A. Let*

$$\mathcal{A} = \left\{a \in A : F\pi(a) - \pi(a)F \in L^1(\mathcal{H})\right\}.$$

(Then $\mathcal{A}$ is a dense subalgebra of A.) Suppose the character ρ on $(\mathcal{H}, \pi, F)$ agrees with τ on $\mathcal{A}$. Then there is no nontrivial idempotent on A.

Note that a character $\rho : \mathcal{A} \to \mathbb{C}$ is $\rho(a) = \frac{1}{2}Trace(\varepsilon F(F\pi(a) - \pi(a)F))$ (Hilbert space trace) .

. (Sketch) The inclusion $\mathcal{A} \hookrightarrow A$ induces an isomorphism:

$$K_0(\mathcal{A}) \to K_0(A).$$

(reason: $\mathcal{A}$ is closed under the holomorphic functional calculus, i.e. if $a \in \mathcal{A}$ and f is holomorphic in an open domain containing the spectrum of A, then

$$f(a) := \int_C \frac{f(z)}{a - z} dz \in \mathcal{A}\,.$$

Therefore, we may assume an idempotent e in A actually lives in $\mathcal{A}$. By K-theory nonsense, we may assume also that $e^* = e$.

From our commutative diagram,

$$\begin{array}{ccccc} Fred(\mathcal{A}) & \times & K_*(\mathcal{A}) & \xrightarrow{\text{index}} & \mathbb{Z} \\ \downarrow^{\text{ch}} & & \downarrow^{\text{ch}} & & \downarrow \\ H_\lambda^*(\mathcal{A}) & \times & H_*^\lambda(\mathcal{A}) & \to & \mathbb{C} \end{array}$$

By hypothesis, we see that $\tau(e) = \rho(e) \in \mathbb{Z}$. We also know that

$$\tau(e) = \tau(e^* e) \geq 0$$

because τ is positive. But $1 - e$ is also a self-adjoint idempotent,

$$\begin{aligned} \tau(1-e) &\geq 0, \\ 1 - \tau(e) &\geq 0 \end{aligned}$$

so $\tau(e) \leq 1$. If $\tau(e) = 0$, then $\tau(e^* e) = 0$ so $\tau(e) = 0$ by faithfulness.

If $\tau(e) = 1$, then $\tau((1-e^*)(1-e)) = 0$, and $1 - e = 0$, $e = 1$. □

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