SMOOTH PROJECTIVE SURFACES

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Abstract. This talk will discuss some basics about smooth projective surfaces, leading to an open question.

1. Surfaces

Definition 1.1. Let k be an algebraically closed field. A **surface** is a two dimensional nonsingular closed subvariety $S \subset \mathbb{P}^n_k$.

Example 1.2. A few examples.

(a) The simplest example is $S = \mathbb{P}^2$.
(b) The nonsingular quadric surfaces $Q \subset \mathbb{P}^3$ with equation $xw - yz = 0$ has Picard group $\mathbb{Z} \oplus \mathbb{Z}$ generated by two opposite rulings.
(c) The zero set of a general homogeneous polynomial $f \in k[x, y, z, w]$ of degree d gives rise to a smooth surface of degree d in $\mathbb{P}^3$.
(d) If C and D are any two nonsingular complete curves, then each is projective (see previous lecture's notes), and composing with the Segre embedding we obtain a closed embedding $S = C \times D \hookrightarrow \mathbb{P}^n$.

1.1. Intersection theory.

Given a surface S, let $\mathrm{Div}S$ be the group of Weil divisors. There is a unique pairing

$$\mathrm{Div}S \times \mathrm{Div}S \to \mathbb{Z}$$

denoted $(C, D) \mapsto C \cdot D$ such that

(a) The pairing is bilinear.
(b) The pairing is symmetric.
(c) If $C, D \subset S$ are smooth curves meeting transversely, then $C \cdot D = \#(C \cap D)$.
(d) If $C_1 \sim C_2$ (they are linearly equivalent), then $C_1 \cdot D = C_2 \cdot D$.

Remark 1.3. When $k = \mathbb{C}$ one can describe the intersection pairing with topology. The divisor C gives rise to the line bundle $L = \mathcal{O}_S(C)$ which has a first Chern class $c_1(L) \in H^2(S, \mathbb{Z})$ and similarly D gives rise to a class $D \in H_2(S, \mathbb{Z})$ by triangulating the components of D as real surfaces. Then $C \cdot D = c_1(L) \cap D \in H^0(S, \mathbb{Z}) \cong \mathbb{Z}$ is simply the cap product.

In view of property (d), the intersection pairing induces a pairing on Picard groups

$$\mathrm{Pic}\, S \times \mathrm{Pic}\, S \to \mathbb{Z}$$

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Example 1.4. When $S = \mathbb{P}^2$, $\operatorname{Pic} S \cong \mathbb{Z}$ is generated by the class L of a line. Since two lines meet in a single (reduced) point, $L \cdot L = 1$. It follows that if $D, E \subset \mathbb{P}^2$ have degrees d, e, then $E \cdot E = de$. This can be thought of as a version of Bezout's theorem.

Example 1.5. The Picard group of the smooth quadric $Q \subset \mathbb{P}^3$ has two free generators L, M in the form of opposite rulings and hence $L \cdot M = 1$. Since the rulings are disjoint in each family we also have $L \cdot L = M \cdot M = 0$, therefore if $(a, b), (c, d) \in \operatorname{Pic} Q$ we have $(a, b) \cdot (c, d) = ad + bc$.

1.2. **The canonical class.** Given a complete surface S as above, there is a closed diagonal embedding $\Delta : S \hookrightarrow S \times S$ whose image has codimension two. If $\mathcal{N}_{S,S\times S}$ denotes the normal bundle to the image $\Delta(S)$, the sheaf of differentials on S is the rank two bundle $\Omega_{S/k} = \Delta^*(\mathcal{N}^\vee_{S,S\times S})$. The canonical line bundle is $\omega_S = \wedge^2 \Omega_{S/k}$, but we write K_S instead when thinking of it as a divisor on S. The geometric genus of S is $\dim_K H^0(\omega_S)$.

1.3. **Adjunction.** If $C \subset S$ is a smooth connected curve, the canonical classes of C and S obey adjunction. In terms of line bundles, this says that

$$\omega_C = \omega_S \otimes \mathcal{O}_S(C)|_C$$

But we can also interpret it in terms of divisors and the intersection pairing: taking degrees gives

$$2g - 2 = (K_S + C) \cdot C$$

Example 1.6. If $C \subset S = \mathbb{P}^2$ is a smooth plane curve of degree d, then $C \sim dL$ for a line L and $K_S = -3L$, so $2g - 2 = (dL - 3L) \cdot dL \Rightarrow 2g - 2 = d(d - 3) \Rightarrow g = \frac{1}{2}(d - 1)(d - 2)$.

Example 1.7. Similarly if $Q \subset \mathbb{P}^3$ is the smooth quadric, then $K_Q = (-2, -2)$ and if $C \subset Q$ is a curve of genus g and type (a, b) as a divisor, then $2g - 2 = (a - 2, b - 2) \cdot (a, b) = 2ab - 2a - 2b$ which explains a result from the previous talks, namely $g = (a - 1)(b - 1)$.

Example 1.8. Let $C \subset \mathbb{P}^3$ be a curve of degree d and genus g and let $C \subset S \subset \mathbb{P}^3$ be a general surface of degree s containing C. Then $\operatorname{Pic} S \cong \mathbb{Z}^2$ is freely generated by $\mathcal{O}_S(C)$ and $\mathcal{O}_S(1)$: I'll just call these C and H. Then the intersection theory on S is given by

(a) $H^2 = s$
(b) $CH = d$
(c) $C^2 = 4d + 2g - 2 - ds$

Parts (a) and (b) come from geometric interpretation of degree. For Part (c), adjunction applied to $S \subset \mathbb{P}^3$ gives $K_S = (s - 4)H$: applying adjunction to $C \subset S$ gives

$$C \cdot (C + (s - 4)H) = C \cdot (C + K_S) = 2g - 2$$

and solving for the self-intersection C^2 gives part (c).

2. Numerical equivalence and the Néron-Severi group

Two divisors E, D on a surface S are numerically equivalent if the intersection pairing cannot tell them apart. More formally, we make the following definition.

Definition 2.1. Two divisors $E, D \in \operatorname{Div} S$ are numerically equivalent if $E \cdot C = D \cdot C$ for each $C \in \operatorname{Div} S$, in which case we write $E \equiv D$. We also let $\operatorname{Div}^0 S \subset \operatorname{Div} S$ be the subgroup of divisors which are numerically equivalent to zero.

Example 2.2. Since the intersection pairing is preserved under linear equivalence, $E \sim D \Rightarrow E \equiv D$, so $\text{Prin} S \subset \operatorname{Div}^0 S$ and we obtain a subgroup $\operatorname{Pic}^0 S \subset \operatorname{Pic} S$ of divisors numerically equivalent to zero. In the previous two examples this is the zero subgroup.

Definition 2.3. The Néron-Severi group of S is $N^1(S) = \operatorname{Div} S / \operatorname{Div}^0 S$.

Remark 2.4. We note a few facts about the Néron-Severi group.

(1) It is a theorem that $N^1(S)$ is a finitely generated free abelian group.
(2) The intersection pairing extends to $N^1(S) \times N^1(S) \to \mathbb{Z}$. BUT we think of N^1 as the line bundles and N_1 as the curves, so this can be written differently.
(3) Over $k = \mathbb{C}$, we have $N^1(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C})$.

3. Positive cones of divisors on a surface

Recall that there are various notions of positivity of a line bundle $L \in \operatorname{Pic} S$: L is very ample if there is a closed embedding $f : S \hookrightarrow \mathbb{P}^n$ with $L = f^*\mathcal{O}(1)$; L is ample if there is $m > 0$ such that $L^{\otimes m}$ is very ample. One can recognize positivity in terms of the intersection pairing.

Theorem 3.1. (Nakai Criterion) *A divisor D on S is ample if and only if $D \cdot D > 0$ and $D \cdot C > 0$ for every irreducible curve $C \subset S$.*

3.1. Real divisors: Nef and ample cones.
Given S, we form the real Euclidean space

$$N^1(S)_{\mathbb{R}} = N^1(S) \otimes_{\mathbb{Z}} \mathbb{R}$$

and think of its elements as Weil divisors $D = \sum c_i C_i$ with $c_i \in \mathbb{R}$. The intersection pairing extends to $N^1(S)_{\mathbb{R}}$, so in view of the Nakai criterion, we'll say that D is ample if $D^2 > 0$ and $D \cdot C > 0$ for each irreducible curve $C \subset S$. Similarly we will say that D is numerically effective if $D \cdot C \geq 0$ for all irreducible curves $C \subset S$.

Definition 3.2. The ample cone $\operatorname{Amp}(S) \subset N^1(S)_{\mathbb{R}}$ is the set of ample $\mathbb{R}$-divisors and the Nef cone $\operatorname{Nef}(S)$ is the set of numerically effective divisors.

Theorem 3.3. *Let S be a surface. Then*

(1) *Both $\operatorname{Amp}(S) \subset N^1(S)_{\mathbb{R}}$ and $\operatorname{Nef}(S) \subset N^1(S)_{\mathbb{R}}$ are cones.*
(2) *$\overline{\operatorname{Amp}(S)} = \operatorname{Nef}(S)$.*
(3) *$\operatorname{Nef}(S)^0 = \operatorname{Amp}(S)$.*

Example 3.4. When $S = \mathbb{P}^2$, $N^1(S)_{\mathbb{R}} \cong \mathbb{R}$ and these cones are just the positive and non-negative reals. When $S = Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, these cones are the strictly positive first quadrant and non-negative first quadrant.

3.2. **Cones of curves.** Thinking of homology classes instead of cohomology classes, we can also form the cone of curves $N_1(S)_{\mathbb{R}} = N^1(S) \otimes_{\mathbb{Z}} \mathbb{R}$. It's the same real vector space as before, but the emphasis is on Weil divisors and curves instead of line bundles. Here there is another cone to consider.

Definition 3.5. The Cone of curves on S is the set $NE(S) \subset N_1(S)_{\mathbb{R}}$ given by

$$NE(S) = \{\sum a_i C_i : 0 \leq a_i \in \mathbb{R} \text{ and } C_i \subset S \text{ is an irreducible curve}\}$$

Topologically you would think of this as a subset of $H_2(S, \mathbb{R})$. Essentially from Nakai's criterion, the cone of curves is related to the ample cone in the expected way:

Theorem 3.6. $\overline{NE}(S) = \{\gamma \in N_1(S)_{\mathbb{R}} : \gamma \cdot D \geq 0$ *for all* $D \in Nef(S)\}$.

4. An open question

Little is known about these cones, though there are examples showing that they can be very complicated when $N^1(S)_{\mathbb{R}}$ has dimension three or more. To make the pictures more tractable, we ask the following question:

Question 4.1. Let $C \subset \mathbb{P}^3$ be a curve and for $d \gg 0$, let $C \subset S \subset \mathbb{P}^3$ be a general surface of large degree. Then $\operatorname{Pic} S \cong \mathbb{Z}^2$ generated by $\mathcal{O}_S(1)$ and $\mathcal{O}_S(C)$. What is $\mathrm{NEF}(S)$?

4.1. **Dolcetti and Ellia (1997).** About twenty years ago Dolcetti and Ellia [2] answered this question for the complete intersection $C = S_m \cap S_n$ of surfaces of degrees $n \leq m$. The result is that the effective cone is bounded by the two rays $\mathbb{R}_+[C]$ and $\mathbb{R}_+[nH - C]$. The calculation is easy and not surprising. General curves are more complicated and there have been no other results in the literature.

4.2. **Chen and Nollet (2015).** Dawei Chen and I worked on this problem for a while after my 2013 visit to Boston University. We tried the simplest non-complete intersection, namely the twisted cubic curve. We obtained the following partial results before we stalled.

Proposition 4.2. *Let I_C be the ideal of the twisted cubic curve $C \subset \mathbb{P}^3$. Then the saturated ideal of the scheme defined by ideal sheaf $\mathcal{I}_C^n$ is precisely I_C^n.*

We thought this was no big deal, but the following summer a preprint of seven authors [1] appeared on the algebraic geometry arXiv where this was the main theorem! They had a more general version for curves with similar minimal resolutions, but the proof was essentially the same. They also gave examples showing that this sort of result is rare.

The NEF cones for smooth quadric and cubic surfaces are well known, so the first interesting case of the question arises when $d = 4$.

Proposition 4.3. *Let $C \subset S$ be a general quartic surface. Then*

(1) *$NE(S)$ is the cone bounded by $\mathbb{R}_+[C]$ and $\mathbb{R}_+[16H - 9C]$.*
(2) *$Amp(S)$ is the cone bounded by $\mathbb{R}_+[2H + 3C]$ and $\mathbb{R}_+[66H - 37C]$.*

The natural "expected" second bounding ray for $NE(S)$ would have been $\mathbb{R}_+[2H - C]$, so the cone of curves is larger than expected, something of a surprise. Our method uses special calculations due to Kovács [4] based on the fact that S is a $K3$ surface and the fact that $(X, Y) = (33, 8)$ is the smallest integer solution to Pell's equation $1 = X^2 - 17Y^2$.

References

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