# THE NOETHER-LEFSCHETZ THEOREM

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## 1. Examples

Earlier we defined the class group Cl$X$ of Weil divisors for an algebraic variety $X$ and the Cartier class group CaCl$X$ of Cartier divisors (which is isomorphic to the Picard group of isomorphism classes of line bundles with tensor product). These groups are isomorphic when $X$ is smooth. In general it is quite difficult to compute these groups. In this section we will give some classic examples without proof.

**Example 1.** Earlier we showed that Cl $\mathbb{C}^n = 0$ and Cl $\mathbb{P}^n \cong \mathbb{Z}$, generated by a hyperplane $H \subset \mathbb{P}^n$.

**Example 2.** A very classical example understood in the 1800s is that of a smooth projective curve $X$. A divisor $D$ on $X$ can be written $\sum n_i p_i$ where $p_i$ are points on $X$, and we can define $\deg D = \sum n_i$. This gives a surjective homomorphism deg : Pic$X \to \mathbb{Z}$ whose kernel consists of the degree 0 divisors, denoted $\text{Pic}^0 X$. Via exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

and the isomorphisms Pic$X \cong H^1(X, \mathcal{O}^*)$ and $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, the degree map can be identified with the cohomology map $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$, so the kernel $\text{Pic}^0 X$ is the quotient $H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$, which shows that $\text{Pic}^0 X$ is an abelian variety (Lie group) of dimension $g$. In particular, if $X$ is not a rational curve (i.e. $g > 0$), then Pic $X$ is not a discrete group.

**Remark 1.** *If $X \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a variety, one can consider the cone $C(X)$ over $X$ in $\mathbb{P}^n$ with vertex $p$. Via the projection map $C(X) \to X$ (whose fibres are lines), one can pull back divisors which gives an isomorphism* Cl $X \to$ Cl $C(X)$.

**Example 3.** The surface $X \subset \mathbb{P}^3$ given by equation $xy - z^2 = 0$ is a cone over the a smooth plane conic (with same equation) in $\mathbb{P}^2$. The plane conic is isomorphic to $\mathbb{P}^1$, so Pic$\mathbb{P}^1 \cong \mathbb{Z}$ is generated by a point by Example 1 and hence Cl $X \cong \mathbb{Z}$ generated by a ruling. This ruling is not a Cartier divisor, but the union of two rulings is (it's a hyperplane section of $X$, see previous talk) and it generates the Pic$X$. Thus Pic $X \subset$ Cl $X$ are both isomorphic to $\mathbb{Z}$ with cokernel $\mathbb{Z}/2\mathbb{Z}$.

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**Remark 2.** *In general Picard groups don't work well with products, but there are two nice special cases:*

*(1)* Pic $(X \times \mathbb{C}^n) \cong$ Pic $X$*, the isomorphism being given by pulling back line bundles under the projection map* $X \times \mathbb{C}_n \to X$*.*

*(2)* Pic $(X \times \mathbb{P}^n) \cong$ Pic $X \oplus \mathbb{Z}$*. Here the projection* $X \times \mathbb{P}^n \to X$ *induces an injection* Pic $X \to$ Pic $(X \times \mathbb{P}^n)$*. One uses the fibres* $\cong \mathbb{P}^n$ *(with Picard group* $\mathbb{Z}$*) to establish the splitting.*

**Example 4.** Consider the smooth quadric surface $X \subset \mathbb{P}^3$ given by equation $xy - zw = 0$. It's not hard to show that $X$ is exactly the image of a closed embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $(a, b), (c, d) \mapsto (ac, bd, ad, bc)$, the Segre embedding. Now Pic $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}$ by Remark 2 above. Moreover, it is generated by opposite rulings on $X$.

**Remark 3.** *It is a general fact that if* $f : \widetilde{X} \to X$ *is the blow-up at a point, then* Pic $\widetilde{X} \cong$ Pic $X \times \mathbb{Z}$*, the new generator being given by the exceptional divisor.*

**Example 5.** If $X \subset \mathbb{P}^3$ is a general cubic surface, it's a rational surface, isomorphic to $\mathbb{P}^2$ with 6 points blown up. Applying Remark 3 successively, we find that Pic $X \cong \mathbb{Z}^7$, generated by the pull-back of a line on $\mathbb{P}^2$ and the 6 exceptional divisors. It is well known that in fact $X$ contains 27 lines.

## 2. Noether-Lefschetz Theorem

If $X \subset \mathbb{P}^n$ is a projective variety and $H \subset \mathbb{P}^n$ is a general hyperplane, one can consider the subvariety $X \cap H \subset X$. There is a restriction map of line bundles $\rho$ : Pic $X \to$ Pic $X \cap H$. We now consider the following general question: when is $\rho$ an isomorphism? Lefschetz proved a result, which was extended by Grothendieck:

**Grothendieck-Lefschetz Theorem.** *Let* $X \subset \mathbb{P}^n$ *be a smooth subvariety and let* $H$ *be a general hyperplane. Then*

$$\rho : \text{Pic } X \to \text{Pic } X \cap H$$

*is an isomorphism if* $\dim X > 3$*.*

**Example 6.** Let $X = \mathbb{P}^n$ for some $n > 3$. One can use the monomials of degree $d$ in the homogeneous coordinates to embed $X$ into a larger projective space $\mathbb{P}^N$; this map is called the $d$-uple embedding $F_d : \mathbb{P}^n \to \mathbb{P}^N$ and the pull-back under $F_d$ of hyperplanes $H \subset \mathbb{P}^N$ gives all the degree $d$ hypersurfaces in $\mathbb{P}^n$. Applying the Grothendieck-Lefschetz theorem, we conclude that for $n > 3$, the general hypersurface $S_d \subset \mathbb{P}^n$ has Pic $S_d \cong \mathbb{Z}$ generated by $H \cap S_d$, where $H$ is a general hyperplane in $\mathbb{P}^n$.

**Question.** *Under what conditions is it true that the restriction map* Pic $\mathbb{P}^n \to$ Pic $S_d$ *is an isomorphism for a general hypersurface* $S_d \subset \mathbb{P}^n$ *of degree* $d$*?*

**Special Cases:** We can answer the question in some special cases fairly easily:

(1) If $n > 3$, the answer is yes by the Grothendieck-Lefschetz theorem.

(2) If $n = 1$, the question is silly because the general hypersurface is a finite set of points, which have trivial Picard group.

(3) If $n = 2$, the answer is yet only if $d = 1$. For $d = 2$ the cokernel is a group of order 2, while for $d > 2$ the hypersurface $S_d$ is a smooth projective curve of genus $g = \frac{1}{2}(d-1)(d-2) > 0$, which has infinitely generated Picard group by Example 2.

(4) The case $n = 3$ is quite interesting. Here we consider different values of $d$:

(a) $d = 1$ the answer is yes.

(b) $d = 2$ the answer is no by Example 4.

(c) $d = 3$ the answer is no by Example 5.

(d) $d \geq 4$ here things are not obvious at all, but Noether had an inspired answer, which is that the answer should be yes.

**Noether's Idea:** The cases $d = 2$ and $d = 3$ fail in large part because general quadric and cubic surfaces necessarily contain LINES. Noether saw that the general QUARTIC equation cannot contain any lines by the following dimension count:

• The space of all quartics is given by their equations modulo scalar. There are 35 monomials of degree 4 in 4 variables, so this family has dimension 34.

• How many quartics contain lines? The family of lines in $\mathbb{P}^3$ has dimension 4, it is given by the Grassmann variety $\mathrm{Grass}_2(4)$. A fixed line $L$ has ideal generated by two linear forms, giving a resolution

$$0 \to S(-2) \to S(-1)^2 \to I_L \to 0$$

from which one can read off $\dim(I_L)_4 = 30$, so modulo scalars there is a 29-dimensional family of quartics containing a fixed line. Adding up, the quartics containing a line form a family of dimension $33 < 34$.

It's hard to extend Noether's idea, because there are way too many families of curves lying on surfaces. However using complex methods and monodromy, Lefschetz [L] was able finish the job:

**Noether-Lefschetz Theorem.** *If $S_d \subset \mathbb{P}^3$ is a general surface of degree $d \geq 4$, then the restriction map* $\mathrm{Pic}\,\mathbb{P}^n \to \mathrm{Pic}\,S_d$ *is an isomorphism.*

**Remark 4.** *In the 1960s, Mumford proposed the challenge of actually writing down a degree $d = 4$ polynomial whose zero set $S_4$ is a smooth surface satisfying the conclusion of the Noether-Lefschetz theorem. This was not achieved until the last few years by Ronald van Luijk* [vL]*. It appears on page 1 of Volume 1 in the new journal "Algebra Number Theory".*

## 3. Recent developments

While the Noether-Lefschetz theorem was proved in the 1920s, there was a revival of interest in the subject around 1990. In 1985 Griffiths and Harris gave a new algebraic proof of the theorem [GH]. There were several new approaches using infinitesimal variations of Hodge structures, and generalizations to singular surfaces. Here's a fun variant of the theorem from Angelo Lopez' 1990 Ph.D. thesis.

**Theorem (Lopez).** *Let $C \subset \mathbb{P}^3$ be a smooth connected curve. If the homogeneous ideal for $C$ is generated by polynomials of degree $\leq d-1$, then the general degree $d$ surface $S_d$ containing $C$ is smooth with Picard group freely generated by the plane $H \cap S_d$ and the divisor $C \subset S_d$.*

The above theorem is appealing because the geometry entirely determines the Picard group. Very recently John Brevik and I extended this result to arbitrary curves in $\mathbb{P}^3$, which may have many components, may have isolated or embedded points, or even by non-reduced in the scheme-theoretic sense [BN]. The specific statement is this:

**Theorem (Brevik and Nollet).** *Let $Z \subset \mathbb{P}^3$ be an arbitrary closed subscheme of dimension $\leq 1$ which lies on a surface with isolated singularities and suppose that the homogeneous ideal of $Z$ is generated by polynomials of degree $\leq d-1$. Then the general degree $d$ surface $S_d$ containing $Z$ is normal with class group freely generated by the plane $H \cap S_d$ and the supports of the curve components of $Z$.*

**Remark 5.** *The theorem above recovers several results in the area, for example:*

- *If $Z = \emptyset$, we recover the original Noether-Lefschetz theorem.*
- *If $Z$ is a smooth connected curve, we recover Lopez' theorem.*
- *If $Z$ is zero dimensional, we recover a theorem of Joshi, which says that the Picard group of the general singular surface is generated by a plane $H$.*

## References


[BN] J. Brevik and S. Nollet, *Noether-Lefschetz theorem with base locus*, preprint (2009).

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[L] S. Lefschetz, *L'analysis situs et la géometrie algébrique*, Gauthier-Villars, Paris, 1924.

[vL] R. van Luijk, *K3 surfaces with Picard number one and infinitelyl many rational points*, Algebra Number Theory **1** (2007), 1–15.