Witten's Proof of Morse Inequalities

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Let M be a smooth, compact, oriented manifold with dimension n. A **Morse function** is a smooth function $f : M \to \mathbb{R}$ such that all of its *critical points* are *nondegenerate*. A point $\overline{x} \in M$ is a **critical point** of f if $df(\overline{x}) = 0$. A critical point is **nondegenerate** if

$$\det(Hess(f)(\overline{x})) = \det\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right](\overline{x}) \neq 0.$$

Both properties do not depend on the choice of coordinates. The index $\operatorname{ind}(\overline{x})$ is the number of negative eigenvalues of $Hess(f)(\overline{x})$. Let $m_p = m_p(f)$ be the number of critical points of index p. Let $b_p = b_p(M) = \dim H^p(M)$ be the dimension of the p^{th} de Rham cohomology group.

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} ... \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

This is called the de Rham complex. Note that $d^2 = 0$. If $\omega = d\alpha$, then $d\omega = 0$. So $\operatorname{Im} d : \Omega^{p-1} \to \Omega^p \subseteq \ker d : \Omega^p \to \Omega^{p+1}$, and

$$H^p(M) = \frac{\ker d}{\operatorname{Im} d}.$$

Theorem 1. *The **Morse inequalities** are as follows. The Morse polynomial $M(t)$ and Poincare polynomial $P(t)$ are defined by*

$$M(t) = \sum_{k=0}^{n} m_k t^k$$
$$P(t) = \sum_{k=0}^{n} b_k t^k.$$

There exists a polynomial $R(t) = R_0 + R_1 t + R_2 t^2 + ...$ with all $R_k \geq 0$ such that

$$M(t) - P(t) = (1 + t) R(t).$$

For example, consider $M =$ the torus with a saddle at the top. Then

$$H^0 = \mathbb{R}, \text{ so } b_0 = 1$$
$$H^1 = \mathbb{R}, \text{ so } b_1 = 2$$
$$H^2 = \mathbb{R}, \text{ so } b_2 = 1.$$

Thus

$$P(t) = 1 + 2t + t^2.$$

Let the Morse function be the height function. Then the critical points are two maxima, three minima, and on minimum. Thus,

$$\begin{aligned} m_0 &= 1 \\ m_1 &= 3 \\ m_2 &= 2 \end{aligned}$$

Thus,

$$M(t) = 1 + 3t + 2t^2.$$

Observe that we have

$$M(t) - P(t) = 1 + 3t + 2t^2 - 1 - 2t - t^2 = t + t^2 = (1+t)\,t.$$

So $R(t) = t$.

Some Corollaries:

Corollary 2. *If* $t = -1$, $M(-1) = P(-1)$, *so* $\sum (-1)^k m_k = \sum (-1)^k b_k = \chi(M) = 2 - 2g$.

In our example, we have $g = 1$, so $0 = 0 = 0 = 0$.

Corollary 3. *We have* $m_k \geq b_k$ *for all* k. *(These are called the weak Morse inequalities.)*

Since $R(t) = \frac{M(t)-P(t)}{1+t} = (M(t) - P(t))\,(1 - t + t^2 - ...)$. Equating the t^k coefficient gives the following.

Corollary 4. *(Strong Morse Inequalities) For each* k, $m_k - m_{k-1} + ... \pm m_0 \geq b_k - b_{k-1} + ... \pm b_0$.

We have the following fact. Suppose that we have

$$0 \to V_0 \xrightarrow{\delta} V_1 \xrightarrow{\delta} ... \xrightarrow{\delta} V_n \xrightarrow{\delta} 0,$$

$\delta^2 = 0$, $\dim V_k < \infty$. If this complex computes the k^{th} cohomology $H^k(M)$ of M, then $\{\dim V_k\}$ obey the Morse inequalities. One example: V_k is a vector space whose basis is the set of critical points of index k. This is Floer cohomology, an important mathematical tool.

We now bring some geometry into the picture. We now put an arbitrary Riemannian metric on the manifold M, giving us a scalar product on the space of differential forms. $(\omega_1, \omega_2) =$ number. We also have an L^2 inner product:

$$\langle \omega_1, \omega_2 \rangle = \int_M (\omega_1, \omega_2)\,.$$

We have $d : \Omega^p \to \Omega^{p+1}$, and the adjoint $d^* : \Omega^{p+1} \to \Omega^p$ is defined by the formula $\langle d\omega, \alpha \rangle = \langle \omega, d\alpha \rangle$ for any p-form ω and any $(p+1)$-form

α. The Laplacian on forms is

$$\Delta = dd^* + d^*d = (d + d^*)^2 .$$

We label $\Delta^p : \Omega^p \to \Omega^p$.

Hodge theory tells us the following. Given a class $[\omega] \in H^p(M)$ (note $[\omega_1] = [\omega_2]$ iff $\omega_1 - \omega_2 = d\alpha$ for some $p-1$-form α). In each class, there is a unique form ω such that $\|\omega\|^2 = \langle \omega, \omega \rangle$ is minimum. This form is the unique harmonic form in the class (i.e. $\Delta_p \omega = 0$).

For example, let $M = S^1$. Identify functions on S^1 with 2π-periodic functions. $H^0(S^1) = \ker d = \{\text{constant functions}\}$. Note these are harmonic, because $\frac{d^2}{d\theta^2}(\text{constant}) = 0$. Next, the one forms ω satisfy $d\omega = 0$. We have to mod out by the image of d on functions. Two one-forms are in the same class if and only if they integrate to the same number. So $H^1(M)$ is generated by $d\theta$. In fact, $d\theta$ is harmonic, since $\frac{d^2}{d\theta^2}(d\theta) = 0$.

The upshot is the following fact.

Corollary 5. *(Hodge)* $\dim \ker \Delta^p = \dim H^p(M) = b_p(M)$.

Motivation: in quantum physics, the eigenvalues of the Laplacian are energies. The energy is zero implies that the particle in the vacuum state. Important: what is the dimension of vacuum space? Same as $\dim \ker(\Delta^p)$. We want to solve $\Delta^p \omega = 0$. The direct method is very difficult. However, the dimension depends only on the topology, due to Hodge theory. Sometimes it is difficult to find the b_p of a very complicated manifold. The next best thing is to estimate b_p using a Morse function, or a vector field, or a one-form.

Witten's idea: given a smooth $f : M \to \mathbb{R}$, the Witten differential is

$$d_s = e^{-sf} d\, e^{sf},$$

where $s \in \mathbb{R}$. Then $d_s^2 = 0$, $ds : \Omega^p \to \Omega^{p+1}$. The Witten Laplacian is

$$\Delta_s^p = d_s d_s^* + d_s^* d_s : \Omega^p \to \Omega^p .$$

We have the following commutative diagram, so e^{-sf} is a quasi-isomorphism, and d_s yields isomorphic cohomology groups.

$$\begin{array}{ccccc} \ldots \to & \Omega^p & \xrightarrow{d} & \Omega^{p+1} & \longrightarrow \ldots \\ & \downarrow e^{-sf} & \circlearrowleft & \downarrow e^{-sf} & \\ \ldots \to & \Omega^p & \xrightarrow{d_s} & \Omega^{p+1} & \longrightarrow \ldots \end{array}$$

So $\ker \Delta_s^p$ is isomorphic to H_s^p, so

$$\dim \ker \Delta_s^p = b_p(M) .$$

We now compute

$$\begin{aligned}
\Delta_s &= d_s d_s^* + d_s^* d_s \\
d_s \omega &= e^{-sf} d\left(e^{sf}\omega\right) \\
&= \left(d + s\ df\wedge\right)\omega \\
d_s^* \omega &= \left(d^* + s\ \nabla f \lrcorner\right)\omega.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta_s &= \left(d + s\ df\wedge\right)\left(d^* + s\ \nabla f\lrcorner\right) + \left(d^* + s\ \nabla f\lrcorner\right)\left(d + s\ df\wedge\right) \\
&= \Delta + s\left(\{df\wedge, d^*\} + \{\nabla f\lrcorner, d\}\right) + s^2\left|\nabla f\right|^2 \\
&= \Delta + s\left(\mathcal{L}_{\nabla f}^* + \mathcal{L}_{\nabla f}\right) + s^2\left|\nabla f\right|^2 \\
&= \Delta + s\left(\mathcal{L}_{\nabla f}^* + \mathcal{L}_{\nabla f}\right) + s^2\left|\nabla f\right|^2
\end{aligned}$$

The first order derivative terms of $\mathcal{L}_{\nabla f}$ and its adjoint are negatives of each other, so that $\mathcal{L}_{\nabla f}^* + \mathcal{L}_{\nabla f}$ is a bounded operator that we call B. Now, suppose that s is very large, and

$$\Delta_s \omega_s = 0.$$

Then

$$\begin{aligned}
0 &= \left\langle \Delta_s \omega_s, \omega_s \right\rangle \\
&= \left|d\omega_s\right|^2 + \left|d^*\omega_s\right|^2 + s\left\langle B\omega_s, \omega_s\right\rangle + s^2\left\langle \left|\nabla f\right|^2 \omega_s, \omega_s\right\rangle.
\end{aligned}$$

The term $\langle B\omega_s, \omega_s\rangle$ is bounded by a constant independent of s. Thus, ω_s must decay rapidly away from the set where $\nabla f = 0$, ie. the critical points of f. So we hope that if f is a nice function, we can approximate it by functions defined only near the critical points. If f is a Morse function, ie. the critical points are nondegenerate, then $\nabla f(\overline{x}) = 0$ implies that the quadratic part of the Taylor expansion is nondegenerate there. If this is the case, then, Δ_s can be approximated by a model operator H. That is, the eigenvalues and eigenforms of the model operator H will be close to the eigenvalues and eigenfunctions of Δ_s. We replace Δ with the second order part with constant coefficients frozen at $\overline{x}$. For example, on functions,

$$\Delta = -\sum \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\frac{\partial}{\partial x^j}\right) \mapsto -g^{ij}\left(\overline{x}\right)\frac{\partial^2}{\partial x^i \partial x^j}.$$

Next, replace B with $B(\overline{x})$. Finally, replace $\left|\nabla f\right|^2$ with its quadratic part at $\overline{x}$.

Theorem 6. *For large s, Δ_s and H have the same number of eigenvalues in the interval $\left[0, s^{-2/5}\right]$.*

Corollary 7. $\dim\ker\left(\Delta_s\right) \leq \#$ $\left(\text{\textit{eigenvalues of } } H \text{\textit{ in} } \left[0, s^{-2/5}\right]\right)$.

Restrict to a single critical point $\overline{x}$. Choose coordinates so that $\overline{x} = 0$, $f(x) = -\sum_{i=1}^{m} (x^i)^2 + \sum_{i=m+1}^{n} (x^i)^2$, index $= m$. Make the metric Euclidean near $\overline{x}$. The model operator is the direct sum of model operators at each critical point. At the critical point $\overline{x}$, the model operator is

$$H_{\overline{x}} = \sum_{i=1}^{n} \left(-\frac{\partial^2}{(\partial x^i)^2} \pm 2s \left[dx^i \wedge, e_i \lrcorner \right] + s^2 x_i^2 \right).$$

These operators summed commute with one another. Note that $-\frac{\partial^2}{(\partial x^i)^2} + s^2 (x^i)^2$ has spectrum $(2+4j)\, s, j \geq 0$ with multiplicity one. The lowest eigenvalue is $\psi_0 = \exp(-sx^2/2)$. Note that $[dx^i \wedge, e_i \lrcorner]$ is

$$\left(dx^i \wedge (e_i \lrcorner) - dx^i \wedge (e_i \lrcorner) \right),$$

which acts by $+1$ if dx^i is in the form, -1 if dx^i is not in the form. So H can have something in the kernel, it can only have an m-form of the form

$$\exp\left(-\frac{s}{2} \sum (x^i)^2 \right) dx^i \wedge ... \wedge dx^m$$

for a critical point of index m. Therefore, for large s,

$$\# \left(\text{eigenvalues of } H \text{ in } \left[0, s^{-2/5} \right] \right) = \# \text{ of critical points of index } p = m_p, \quad \text{so } b_p \leq m_p .$$