MORSE THEORY

1. Introduction to Morse Theory

Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map. This induces $DF_x : T_xM \to T_{f(x)}N$.

Definition 1.1. *$x \in M$ is* ***critical*** *for F if $DF_x < \min\{\dim M, \dim N\}$. Otherwise, x is* ***regular****.*

Cr_F = the set of critical points of F.

Definition 1.2. *$y \in N$ is a critical value if $F^{-1}(y)$ contains a critical point. Otherwise it is a regular value.*

Δ_F =the set of critical values of F = ***Discriminant set.***

Theorem 1.3. *(Morse-Sard-Federer Theorem)*

(1) *Δ_F has measure 0.*
(2) *If $F(M)$ has nonempty interior, then the set of regular values is dense in the image $F(M)$.*

Examples

(1) Standard $N = \mathbb{R}$. $x \in M$ is critical implies $df_x = 0$.
(2) Torus maps to height function, 4 critical points.
(3) horizontal torus maps via height function, 2 critical points, union of two circles (top and bottom) as critical sets.
(4) If $M \subset \mathbb{R}^n$, then $f : \mathbb{R}^n \to \mathbb{R}$ is critical at p on M if its tangent space T_pM is tangent to a level set (in $\mathbb{R}^n$) of f.
(5) If F is proper analytic between analytic manifolds, then Δ_F is a union of submanifolds of N.

Recall if M is a manifold and X is a vector field on M and f is a smooth (real-valued) function, $X(f) = df(X)$.

Lemma 1.4. *If p_0 is a critical point of f, and of X, X', Y, Y' are vector fields such that*

$$X(p_0) = X'(p_0),\ Y(p_0) = Y'(p_0),$$

then

$$(X(Yf))(p_0) = (X'(Y'f))(p_0) = (Y(Xf))(p_0)$$

(not usually true).

Proof. $[(XY - YX)f](p_0) = ([X, Y]f)(p_0) = df([X, Y])(p_0) = 0$.

Since $((X - X')f)(p_0) = 0$, etc. the result follows. □

Definition 1.5. *If p_0 is a critical point for $f : M \to \mathbb{R}$, define the Hessian as $H_{f,p_0} : T_{p_0}M \times T_{p_0}M \to \mathbb{R}$, where*

$$H_{f,p_0}(X_0, Y_0) = (XYf)(p_0)$$

such that $X(p_0) = X_0$, $Y(p_0) = Y_0$. This is well-defined and symmetric, by the Lemma.

If $X=\sum X^j\partial_j$, $Y=\sum Y^i\partial_i$ in local coordinates at p_0,

$$\begin{aligned}H_{f,p_0}(X_0,Y_0) &= \sum h_{ij}X^iX^j\\ h_{ij} &= \partial_i\partial_j f(p_0).\end{aligned}$$

Definition 1.6. *p_0 is **nondegenerate** if H is nondegenerate, ie $H(X,Y)=0$ for all Y implies $X=0$.*

If so, $f(x)=f(p_0)+\frac{1}{2}H_{p_0}(x,x)+\mathcal{O}\left(|x|^3\right)$.

Definition 1.7. *f is called a **Morse function** if all critical points are nondegenerate.*

Note that H is nondegenerate iff $\det(h_{ij})\neq 0$. For example, $f(x)=x^3$ from $\mathbb{R}$ to $\mathbb{R}$ has $H=0$ at $x=0$. The "lying down" torus has degenerate critical points (H rank 1 instead of 2).

Given a symmetric bilinear map $\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, there is a basis with respect to which it's diagonal. Nondegenerate implies the matrix is full rank. And the number of negative eigenvalues is called the **index** of the critical point. This is also the largest dimension of a subspace on which H is negative definite. Let

$$\lambda(f,p_0)=\text{index of } f \text{ at } p_0.$$

Define the **Morse polynomial**

$$P_F(t)=\sum_{p\in C_{rF}} t^{\lambda(f,p)}=\sum_{\lambda\in\mathbb{Z}}\mu_f(\lambda)\,t^\lambda$$

where $\mu_f(\lambda)$ is the number of critical points with index λ.

Theorem 1.8. *("Morse lemma") If f is Morse and p is a nondegenerate critical point, there exists a neighborhood U of p and coordinates such that $x^i(p)=0$ and such that $f(x)=f(p_0)+\frac{1}{2}H_{f,p}(x)$.*

Corollary 1.9. *There exist coordinates such that*

$$f(x)=f(p_0)-\sum_{j=1}^{\lambda}\left(x^i\right)^2+\sum_{j=\lambda+1}^{m}\left(x^i\right)^2$$

These are called coordinates adapted to f.

2. Existence of Morse Functions

Assume M is imbedded in $E=\mathbb{R}^{2m+1}$, with some metric. Let Λ be a smooth manifold. Suppose $F:\Lambda\times E\to\mathbb{R}$ is smooth, which restricts to $f:\Lambda\times M\to\mathbb{R}$. Let $f_\lambda:\lambda\times M\to\mathbb{R}$ be the family of functions. There is a natural surjection by restriction:

$$p_x:E^*\to T_x^*M..$$

So $df_\lambda(x)=PdF_\lambda(x)$. Let

$$\begin{aligned}\partial^x f &: \Lambda\to T_x^*M\\ \lambda &\mapsto df_\lambda(x).\end{aligned}$$

Definition 2.1. *We say $F:\Lambda\times E\to\mathbb{R}$ is **sufficiently large relative to M** if* $\dim\Lambda\geq\dim M$ *and $\forall x\in M$, the point $0\in T_x^*M$ is a regular value for $\partial^* f$.*

Theorem 2.2. *If F is sufficiently large, then there exists a subset $\Lambda_\infty \subset \Lambda$ of measure zero such that f_λ is Morse for all $\lambda \in \Lambda - \Lambda_\infty$.*

Example: let $\Lambda = E^*$, let $H = E^* \times E \to \mathbb{R}$. (evaluation). Claim : this is sufficiently large. This is the height function.

Example: Let $\Lambda = E$, let $F = R : E \times E \to \mathbb{R}$, $R(\lambda, x) = \frac{1}{2} |x - \lambda|^2$. This R is sufficiently large. The resulting Morse functions are **exhaustive**, ie the sublevel sets $\{x \in M : f(x) \leq s\}$ are compact (assuming M is properly embedded in E, equivalent to M being embedded in E as a closed subspace).

Example: Let Λ be the set of positive definite matrices on E . Let $F : \Lambda \to E$ be $(A, x) \mapsto \frac{1}{2}(Ax, x)$. This F is sufficiently large and exhaustive.

The point: there exist Morse functions.

Definition 2.3. *A Morse function is called **resonant** if two critical points have the same critical value.*

Theorem 2.4. *Any resonant Morse function on a compact manifold can be approximated arbitrarily closely in the C^2 topology by a nonresonant Morse function.*

3. The topology of Morse functions

3.1. **Surgery, handle attachment, cobordism.** Let D^k be the closed k-disk. Let $\mathring{D}^k$ denote the interior, $\partial D^k = S^{k-1}$ be the boundary.

Given X and a map from $\partial D^k \to X$, we can do a handle attachment (cell attachment). A good way to build CW complexes. But this bad in the manifold world.

Instead, we will do surgery (attaching handles one dimension up). Let M be a smooth manifold, given an embedding of a sphere into M and a trivial normal bundle that extends the embedding. (more next time)

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