INTRODUCTION TO THE WORK OF MIRIAM MIRZAKHANI

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Contents

1. Riemann surfaces and uniformization

Let S be a Riemann surface, i.e. a connected Hausdorff space such that a space of continuous, complex-valued functions defined on subdomains of S is designated as the space of **holomorphic** functions, and such that

(1) for every point $p \in S$, there exists a neighborhood U of p and a holomorphic function $\zeta : U \to \mathbb{C}$ that is a homeomorphism onto its image, which is a domain in $\mathbb{C}$; these coordinate charts are called **local parameters.**
(2) the transition functions $\phi_U \circ \phi_V^{-1}$ are holomorphic where defined.
(3) If $D \subseteq S$ is a domain and $f : D \to \mathbb{C}$ is a function, then f is holomorphic if and only if for every local parameter ζ defined in any $U \subseteq D$, the function $f \circ \zeta^{-1}$ is holomorphic in the ordinary sense on $\zeta(U) \subseteq \mathbb{C}$.

It follows that S is a surface (i.e. locally homeomorphic to $\mathbb{R}^2$), it is orientable, and it is triangulable (the last one is not so obvious).

Examples of Riemann surfaces include $\mathbb{C}$, the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ [z is a local parameter on $U = \mathbb{C}$; $\zeta = \frac{1}{z}$ is a local parameter on $\widehat{\mathbb{C}} \setminus \{0\}$], every domain in $\mathbb{C}$, eg. the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, any subdomain of a Riemann surface. Tori, hyperbolic surfaces, etc.

A homeomorphism h between two Riemann surfaces S_1 and S_2 is called **conformal** if h and h^{-1} take (germs of) holomorphic functions to (germs of) holomorphic functions. Note that one may use a local parameter to measure angles on a Riemann surface. Conformal maps are exactly the angle-preserving and orientation-preserving maps from domains in $\mathbb{R}^2$ to other domains in $\mathbb{R}^2$.

The ring of holomorphic functions on a noncompact Riemann surface and the field of meromorphic functions on any Riemann surface each determine the Riemann surface uniquely, up to a conformal mapping and a reflection.

Gauss proved in 1822 that at any point on any sufficiently smooth oriented surface Σ in Euclidean space with the induced metric, there exists a neighborhood which can be mapped conformally onto a plane domain. (In fact, he showed that any metric on the surface is conformal to the Euclidean metric.) Therefore, any such Σ can be made into a Riemann surface by declaring a nonconstant continuous complex-valued function $g : D \to \mathbb{C}$ on a domain $D \subseteq \Sigma$ to be holomorphic if it is locally, except at isolated points, a conformal mapping.

Here is the idea of Gauss' proof. It turns out the embedding is irrelevant. All that matters is the existence of a Riemannian metric

$$ds^2 = E(x, y)\, dx^2 + 2F(x, y)\, dxdy + G(x, y)\, dy^2.$$

A local parameter on a part of Σ is a complex-valued function $\zeta = u(x, y) + iv(x, y)$ whose Jacobian is positive and such that the metric is

$$ds^2 = \rho(u, v)\left(du^2 + dv^2\right);$$

the real functions u, v are called **isothermal coordinates**. To find these, do the following: choose a local basis of eigenvectors of the matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$; these will be vector fields that are linearly independent in a neighborhood of the point in question. Consider the integral curves of these two vector fields to be level sets of the functions u and v. By choosing one integral curve of the first vector field, we can choose any increasing function along that curve, and its values would define the values of u on the whole neighborhood (by extending the function to be constant on the integral curves of the other vector field). By switching the roles of the two vector fields, we may define the function v. Then the given metric in these coordinates has the form $\begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$. If we change variables again, we can make $E = G$. But we realize that we run into trouble with this approach if there is one eigenvalue of multiplicity two.

What Gauss proved is that if E, F, G are real analytic functions of (x, y), then the isothermal coordinates (u, v) exist. Since then mathematicians have found that you can weaken the hypothesis (eg Morrey showed in 1938 that it is enough to assume E, F, G are measurable and $\frac{E+G}{\sqrt{EG-F^2}}$ is bounded. Here is the rough idea of how this is done.

Given that

$$ds^2 = E(x,y)\,dx^2 + 2F(x,y)\,dxdy + G(x,y)\,dy^2,$$

we write in complex coordinates $z = x + iy$ to get

$$ds^2 = \lambda\,|dz + \mu d\overline{z}|^2,$$

where

$$\begin{aligned} \lambda &= \frac{1}{4}\left(E + G + 2\sqrt{EG - F^2}\right), \\ \mu &= \frac{E - G + 2iF}{4\lambda}. \end{aligned}$$

The isothermal coordinate case would be the case where $F = 0$, $E = G$, so $\lambda = E$, $\mu = 0$. So we wish to find new coordinates u, v, $w = u + iv$ where

$$\begin{aligned} ds^2 &= \rho\left(dx^2 + dy^2\right) \\ &= \rho\,|dw|^2 \\ &= \rho\,|w_z|^2 \left|dz + \frac{w_{\overline{z}}}{w_z} d\overline{z}\right|^2 \quad \text{(in old coords)} \end{aligned}$$

So it turns out that we can find (u, v) isothermal if and only if we can find a complex diffeomorphism $z \mapsto w(z)$ such that

$$\frac{\partial w}{\partial \overline{z}} = \mu \frac{\partial w}{\partial z}.$$

It can be shown that there exists a solution to this equation whenever $\|\mu\|_\infty < 1$.

Another geometric way to show that you can get isothermal coordinates is using the Hodge star operator. Note that (u, v) is isothermal if and only if $*du = dv$ [see below for explanation (1)]. By standard elliptic theory, u can be chosen to be harmonic ($\Delta u = 0$) near a given point, with du nonvanishing [see below for explanation (2)]. By the Poincaré lemma, $*du = dv$ has a local solution v exactly when $d * du = 0$, ie when $- * d * du = \Delta u = 0$. Since du is nonzero, $*^2 = -1$ on one-forms, we have du and dv are necessarily linearly independent, so we get isothermal coordinates!

[Explanations of details:

(1) The Hodge star operator $*$ is defined on p-forms on a Riemannian manifold M by the equation $\alpha \wedge *\beta = (\alpha, \beta)\, dV$, where α and β are any p-forms and dV is the volume form, and $(\bullet , \bullet)$ is the pointwise inner product of forms. This pointwise inner product is defined using the Riemannian metric; if $e_1, ..., e_n$ is a local orthonormal frame of the tangent bundle TM, then $\left\{ e^*_{i_1} \wedge ... \wedge e^*_{i_p} \right\}$ is declared to be a local orthonormal frame of $\Lambda^p T^* M$ and thereby determines the metric on $\Lambda^p T^* M$. For the particular case of 1-forms, it turns out that if $g_{jk} := \langle \partial_{x^j}, \partial_{x^k} \rangle$ then $\left(dx^j, dx^k \right) = g^{jk}$, where $\left(g^{jk} \right)$ is the inverse matrix of (g_{jk}). Also, the volume form in any coordinates is $dV = \sqrt{g} dx_1 \wedge ... \wedge dx_n$, where $g = \det(g_{ij})$. So if we compute the Hodge star operator on 1-forms on a 2-manifold, we get for example that $dx^1 \wedge *dx^1 = (dx^1, dx^1)\, dV$, so $dx^1 \wedge *dx^1 = g^{11}\sqrt{g} dx^1 \wedge dx^2$, and similarly $dx^2 \wedge *dx^1 = g^{21}\sqrt{g} dx^1 \wedge dx^2$, from which we can deduce that $*dx^1 = -g^{21}\sqrt{g} dx^1 + g^{11}\sqrt{g} dx^2$. Similarly, $*dx^2 = -g^{12}\sqrt{g} dx^2 + g^{22}\sqrt{g} dx^1$. So if $*dx^1 = dx^2$, then $g^{21} = g^{12} = 0$ and $g^{11}\sqrt{g} = 1$, so that $\frac{1}{g_{11}}\sqrt{g_{11} g_{22}} = 1$, so $g_{11} = g_{22}$. The converse is also true.

(2) Since Δ is elliptic, it has a unique solution to the Dirichlet problem. That is, given a bounded region with reasonable boundary and a function on the boundary, there is a unique harmonic ($\Delta u = 0$) function that has those boundary values. So by choosing a small neighborhood around the origin of the coordinate system, we may set the boundary values so that the value of the function is cx^1 on the boundary, with c large. By solving the Dirichlet problem at taking c to $+\infty$, we can guarantee that the solution, which must be close to cx^1, must satisfy $du \neq 0$ at the origin for some large c, and thus it satisfies $du \neq 0$ nearby.

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Note that the Gauss curvature in isothermal coordinates is

$$K = -\frac{1}{2\rho} \left(\frac{\partial}{\partial u} \left(\frac{1}{\rho} \frac{\partial \rho}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\rho} \frac{\partial \rho}{\partial v} \right) \right),$$

So, for instance, if $\rho = \frac{1}{v^2}$, then

$$\begin{aligned} K &= -\frac{1}{2}v^2 \frac{\partial}{\partial v}\left(v^2\left(-2v^{-3}\right)\right) \\ &= v^2 \frac{\partial}{\partial v}\left(v^{-1}\right) = -1. \end{aligned}$$

Also, the Laplacian in these coordinates is

$$\begin{aligned} \Delta &= -\frac{1}{\sqrt{g}}\partial_j\left(g^{ij}\sqrt{g}\partial_i\right) \\ &= -\frac{1}{\rho}\left(\partial_u^2 + \partial_v^2\right). \end{aligned}$$

So, every Riemann surface can be defined by putting a Riemannian metric onto an orientable, sufficiently smooth surface. Surprisingly, one may require that this metric is complete (ie.every geodesic can be extended indefinitely) and has constant Gauss curvature $+1$, -1, or 0. How does one find this metric?

The **uniformization theorem** proved by Klein, Poincaré, Koebe in 1882-1907 states that every simply connected Riemann surface is conformal to one of

$$\widehat{\mathbb{C}},\ \mathbb{C}, \mathbb{H}.$$

Then , for any Riemann surface S, we have $S = \widetilde{S}\diagup G$ where $\widetilde{S}$ is the universal cover and G is the fundamental group. The universal cover $\widetilde{S}$ can be made into a Riemann surface by lifting the local parameters. Note then that the deck transformations act by conformal maps. Then $\widetilde{S}$ must be conformal to one of the three surfaces above, and the covering transformation group G is a discrete, fixed point free group of conformal mappings of $\widetilde{S}$ to itself, i.e. conformal automorphisms. In the particular case of $\widetilde{S} = \mathbb{C}$, the only differentiable holomorphic maps are rational functions. But in order for the map to be a bijection, it must have only one zero and one pole, so it must be a discrete subgroup of Möbius transformations

$$z \mapsto \frac{az+b}{cz+d},\ ad - bc = 1.$$

If $\widetilde{S} = \widehat{\mathbb{C}}$, G can only contain the identity (because no nontrivial quotients of the sphere are oriented surfaces), so S is the Riemann sphere.

If $\widetilde{S} = \mathbb{C}$, then G must be a discrete subgroup of the (orientation-preserving) conformal mappings of $\mathbb{C}$ to itself, which are actually the subgroup of Möbius transformations that fix infinity, so $c = 0$ and they are of the form $z \mapsto az + b$. But the only way to get a discrete, fixed point free subgroup out of this is to have $a = d = 1$, so the group

G can only contain Euclidean translations. In this case, we can show that either G is trivial (S is conformal to $\mathbb{C}$), G is generated by one translation (say by $z \mapsto z+1$, and S is conformal to $\mathbb{C} \setminus \{0\}$), or G is generated by two linearly independent translations (say $z \mapsto z+1$ and $z \mapsto z+\tau$ with $\mathrm{Im}\tau > 0$, in which case S is conformal to a torus). Observe that we may always rotate and rescale so that one of the generating translations is $z \mapsto z+1$.

If $\widetilde{S}$ is $\mathbb{H}$, then G is a discrete group of Möbius transformations such that $a, b, c, d \in \mathbb{R}$. Such groups are called **Fuchsian groups**. For a Riemann surface, the Fuchsian group G must be fixed point free.

Now, $\widehat{\mathbb{C}}$ carries a complete metric of Gauss curvature 1 (pulled back from the standard $S^2 \subseteq \mathbb{R}^3$ by the iinverse of stereographic projection), $\mathbb{C}$ has a complete metric of curvature 0, and $\mathbb{H}$ has the complete metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

In each case, the group G acts by isometries.

Surfaces of the form $\widehat{\mathbb{C}} \diagup G$ carry a (complete) metric of constant Gauss curvature -1, and these are called **hyperbolic surfaces**. Note that by the Gauss-Bonnet Theorem, if the area of the hyperbolic surface is finite, then

$$\int_{\widehat{\mathbb{C}} \diagup G} (-1)\, dA = 2\pi\chi \left(\widehat{\mathbb{C}} \diagup G\right).$$

where $\chi \left(\widehat{\mathbb{C}} \diagup G\right)$ is the Euler characteristic. Therefore, hyperbolic surfaces with finite area must have negative Euler characteristics. Therefore, among closed orientable surfaces, the sphere carries a metric of constant curvature $+1$, the torus carries a metric of constant curvature 0, and genus g surfaces with $g > 1$ carry a metric of constant curvature -1, since $\chi = 2 - 2g$. See an Escher print to see the fundamental domains of the genus g surfaces inside the upper half plane or Poincaré disk. Further, any given metric on these surfaces must be globally conformal to one of those specific constant curvature metrics.

When the surface is noncompact, the Euler characteristic is not enough to determine whether the surface is hyperbolic or flat (note that the metric cannot have constant curvature $+1$ because S^2 can not have noncompact quotients). For example, consider these noncompact Riemann surfaces (use the standard metric in $\mathbb{C} = \mathbb{R}^2$ to start out).

$$\begin{aligned} S_{a,b} &= \{z \in \mathbb{C} : a < |z| < b\}\,,\ a, b \in (0, \infty) \\ S_0 &= \mathbb{C} \setminus \{0\} \end{aligned}$$

Both of these surfaces have Euler characteristic 0 (both homotopic to the circle). Observe that $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a map that is a local (conformal) diffeomorphism (with local inverse $w = \log(z)$ with some branch). Therefore, this is the universal covering map, and the (flat) metric upstairs can be pushed down to a metric on the punctured plane. Therefore, the standard metric on the punctured plane S_0 is conformal to the flat Euclidean metric (which makes this a metric cylinder). Specifically, observe that $ds^2 = |dw|^2 = \frac{1}{|z|^2}|dz|^2$ is the flat metric on the punctured plane, conformal to the original metric $|dz|^2$. As a consequence, we see that by using the the stereographic projection that the sphere with two punctures is conformal to this same complete flat metric, that of the infinite flat cylinder. On the other hand, observe that we can use the following sequence of holomorphic maps to go from the upper half plane $\mathbb{H}$ to $S_{a,b}$. First, map the upper half plane to $\{z : 0 < \text{Im}z < \pi\}$ via the standard $\log(z)$. Then multiply by $-i$ to get a vertical strip. Then rescale by multiplying by a positive constant and adding a positive real number to map this to the strip $\{z : \log a < \text{Re}z < \log b\}$. So far all maps have been conformal diffeomorphisms. Now map by the conformal covering map $z \mapsto \exp(z)$ to get $S_{a,b}$. Thus, we may pullback the hyperbolic metric on $\mathbb{H}$ to get a complete hyperbolic metric on $S_{a,b}$. So, S_0 is a flat Riemann surface, and $S_{a,b}$ is a hyperbolic Riemann surface.

Note that there is another way to tell that the complete metric for $S_{a,b}$ that is conformal to the original metric must have negative curvature and thus must be conformal to a curvature -1 metric. First, given any conformal metric $ds^2 = \lambda(z)|dz|^2$, we may average over rotations so that $\lambda(z)$ is rotationally symmetric, so that λ depends only on $|z|$. Then, observe that since curves approaching both $|z| = a$ and $|z| = b$ must be infinite in length, $\lambda(z) \to \infty$ as $|z| \to a$ or $|z| \to b$. But then this means that since $|dz|^2 = dr^2 + r^2 d\theta^2$ in polar coordinates that the circles $|z| =$ constant have lengths that approach infinity as $|z| \to a$ or b. This cannot happen on the cylinder. Also, for the circle $|z| =$ constant which has minimum length, the lengths increase on either side, so that the Gauss curvature must be negative. Observe that this reasoning does not work for the punctured plane, because then the polar metric has $r \to 0$ near the puncture, and the circles around the origin to not have to get to infinite length. Using this same reasoning, we see that a sphere with 3 or more punctures must be a Riemann surface of hyperbolic type. Similarly, a plane with two or more punctures must be of hyperbolic type. A flat torus with one or more punctures has a covers with at least three punctures, so using analogous reasoning we can show that the complete conformal metric for this one is hyperbolic. Likewise, a hyperbolic surface with any number of punctures is conformal to a complete metric of constant curvature -1.

2. Geometry of hyperbolic surfaces

The following results about hyperbolic surfaces will be useful in the future.

Observe that the infinite geodesics in the upper half plane $\mathbb{H}$ are either vertical lines or semicircular arcs whose center is on the real axis.

Theorem 1. *Let S be a compact hyperbolic surface, and let c be a closed curve in S that is not homotopic to a constant. Then*

1. *c is freely homotopic to a unique closed geodesic γ.*
2. *Either $\gamma \subseteq \partial S$ or $\gamma \cap \partial S = \varnothing$.*
3. *if c is simple, then γ is simple.*
4. *if c is a smooth boundary component of S, then γ and c bounded an imbedded annulus.*

Proposition 2. *Given c, c' closed curves on a compact hyperbolic surface such that they are not homotopic to constants and intersect each other in n points (counted with multiplicity), then the corresponding closed geodesics γ and γ' satisfy either $\gamma = \gamma'$ (as points) or γ and γ' intersect in at most n points.*

Lemma 3. *For any positive real numbers a, b, c there exists a right-angled geodesic hexagon in $\mathbb{H}$ with nonadjacent side lengths a, b, c. The hexagon is unique up to isometry.*

Proof. Start with a finite vertical geodesic β with $\mathrm{Re}\,(\beta\,(t)) = 0$, and let α be a geodesic in the first quadrant perpendicular to β at its lower endpoint, so that it is a semicircular arc centered at 0. Let δ be a third geodesic that is perpendicular to α at its other endpoint so that the length of α is a. Let $\ell_m = \{x + imx : x > 0\}$ be the set of points in $\mathbb{H}$ such that the distance from β to ℓ_m is c. Let γ be the unique geodesic that is tangent to ℓ_m and such that its distance from δ is b. We realize that distance with a geodesic. Then we connect the point of tangency with the first curve β to obtain a perpendicular geodesic of length c. The resulting hexagon has the desired properties. The hyperbolic trigonometry (laws of sinhs and coshs) forces the other three side lengths to be determined by a, b, c, thus yielding uniqueness. □

We may now paste two copies of a right-angled hexagon as above to make a **pair of pants** (or Y **piece**), which is a sphere with three boundary components of length $2a$, $2b$, $2c$. In general, a pair of pants is a sphere with three boundary geodesic components.

Proposition 4. *For any positive real numbers ℓ_1, ℓ_2, ℓ_3, there exists a unique pair of pants Y with boundary geodesics α, β, γ of lengths ℓ_1, ℓ_2, ℓ_3, respectively.*

Proof. Existence has already been shown. Given any Y-piece, there exists a unique simple perpendicular geodesic that connects every pair of boundary geodesics. Cutting along these three geodesics decomposes the Y-piece into two isometric right angled hexagons. The lemma can be used to show uniqueness. □

Next, we can combine pairs of pants together to create a hyperbolic surface by using a **cubic graph** G. Such a graph is a finite, 3-regular connected graph. Each vertex will correspond to a pair of pants, and each edge will correspond to the gluing of the two boundary components.The number $V(G)$ of vertices of G is always an even number, so we will write it as $V(G) = 2g - 2$. The number of edges is then $\frac{3}{2}$ times as many, so there will be $E(G) = 3g - 3$ edges. Let $\{y_1, ..., y_{2g-2}\}$ be the vertices of G, and let $\{c_1, ..., c_{3g-3}\}$ be the edges. We divide each edge in two, and we number them as $c_{i\alpha}$ for the c_{i1}, c_{i2}, c_{i3} as the three half-edges emanating from y_i . We write for example $c_k = (c_{i\mu}, c_{j\nu})$ for the edge that connects y_i to y_j. A list $\{(c_{i\mu}, c_{j\nu})\}$ is called **admissible** if each symbol occurs exactly once, and we identify y_i with $\{c_{i1}, c_{i2}, c_{i3}\}$. Thus the admissible lists correspond exactly to **marked cubic graphs**.

Two Y-pieces Y, Y' may be glued along a geodesic γ, γ' boundary component of length ℓ via the map

$$\gamma(t) = \gamma'(\alpha - t),$$

where $\alpha, t \in \mathbb{R}$. The number α is called the **twist parameter**. Therefore, we may construction the various Riemann surfaces $F(G, L, A)$, where $G = \{(c_{i\mu}, c_{j\nu})\}$, $L = \{\ell_1, ..., \ell_{3g-3}\}$, $A = \{\alpha_1, ..., \alpha_{3g-3}\}$, where the indices within L and A correspond to the edges of G, with L giving the lengths of the geodesic boundary components of the Y-pieces and A giving the twist parameters for gluing the Y pieces together.

3. Generalizations

If G is any Fuchsian group (ie even one with torsion and fixed points), then $S = \mathbb{H}/G$ is also a Riemann surface in a natural way. The holomorphic and meromorphic functions on S are simply the G-invariant (or **automorphic**) functions on $\mathbb{H}$. Similarly, for every integer q, the holomorphic and meromorphic q-differentials on S are the holomorphic and meromorphic functions φ on $\mathbb{H}$ such that $\varphi(x)\,dz^q$ is G-invariant, meaning that

$$\varphi(gz)\,(g'(z))^q = \varphi(z)$$

for all $g \in G$. These are called **automorphic forms**, which can be defined as sections of line bundles over S. A special convention is needed at elliptic and parabolic fixed points (see Igor's lecture :)).

Fuchsian groups are special cases of **Kleinian groups**, which are Möbius transformations with a, b, c, d not necessarily real, with the following property. The **limit set** Λ of G is the set of limit points of $T_z = \{g(z) : g \in G\}$ for different $z \in \mathbb{C}$. The group is called Kleinian if Λ is not all of $\widehat{\mathbb{C}}$. In this case, the open dense set $\Omega = \widehat{\mathbb{C}} \setminus \Lambda$ is called the **region of discontinuity** of G, and every component of Ω is called a component of G. The group G acts properly discontinuously on Ω, and the quotient $\Omega \diagup G$ is a disjoint union of Riemann surfaces. Ahlfors proved that if a Kleinian group G is finitely generated, the quotient $\Omega \diagup G$ has finitely many components, and each component is a compact Riemann surface or a compact Riemann surface minus a finite number of points. Further, the projection $\Omega \to \Omega \diagup G$ is ramified over at most finitely many points.

4. Algebraic Curves

Let S be a compact Riemann surface. The only holomorphic functions are constants, but the meromorphic functions form a field of algebraic functions of one variable. This means two meromorphic functions, z and w, are connected by an irreducible polynomial equation with complex coefficients

$$P(z, w) = \sum_{\nu=1}^{n} \sum_{\mu=1}^{m} a_{\nu\mu} z^\nu w^\mu = 0.$$

Further, if the z and w are suitably chosen, any other meromorphic function is a rational function of z and w. If so, we say that S is the Riemann surface of the plane algebraic curve $P(z, w) = 0$. This does not mean that S is isomorphic to the zero set of P, or even the homogeneous 3-variable version of this in $\mathbb{CP}^2$. In general the curve $P(z, w) = 0$ will have singularities. However, it is true that every compact Riemann surface is isomorphic to a nonsingular algebraic curve in $\mathbb{CP}^3$.

5. noncompact Riemann surface types

A (noncompact) Riemann surface S with finitely generated fundamental group can always be obtained from a compact Riemann surface $\widehat{S}$, of some genus g by removing $r > 0$ disjoint continua (nonempty compact connected metric spaces). If n of those are points and $m = r - n$ are nondegenerate continua, we say S has type (g, n, m), or type (g, n) if $m = 0$.

A Riemann surface of type (p, n, m) with $m > 0$ is said to have m **ideal boundary curves**. Such an S can always be doubled across the boundary to obtain a surface S^d of type $(2g + m - 1, 2n, 0)$, which admits an anticonformal involution j that fixes the ideal boundary curves $C_1, ..., C_m$. The surface S^d is the **Schottky double** of S. For example, the Schottky double of $\mathbb{H}$ is $\widehat{\mathbb{C}}$.

6. Moduli spaces

By the uniformization theorem, every compact Riemann surface of genus 0 is conformal to a sphere ($\widehat{\mathbb{C}}$). The conformal type of a compact surface of genus 1 depends on one complex parameter $\tau \in \mathbb{H}$. Riemann computed that the conformal type of a compact Riemann surface of genus $g > 1$ depends on $3g - 3$ complex parameters (**moduli**). This implies that the number of complex parameters for the conformal type of a surface of type (g, n) is $3g - 3 + n$. Similarly, the conformal type of a surface of type (g, n, m) with $m > 0$ depends on $6p - 6 + 2n + 3m$ real parameters.

The main aim of Teichmüller theory is to make the dependence of a Riemann surface of finite type on the complex or real moduli as explicit as possible. He recognized that the problem becomes more accessible if we consider more general quasiconformal mappings between Riemann surfaces.

There are no nontrivial quotients of $S^2 = \widehat{\mathbb{C}}$ that are orientable surfaces, so this is the only manifold and conformal class of a surface of genus 0.

Given a compact surface of genus 1, it is a torus with universal cover $\mathbb{C}$, so there exists a metric of constant curvature 0 in its conformal class. As explained earlier, the deck transformations must be conformal transformations, and the only possibility is that the deck transformation group is generated by two independent translations of the plane, i.e. two vectors in the plane. By a rotation and dilation, we can assume the first vector is the number 1, and the second vector is a complex number τ with $\mathrm{Im}\tau > 0$, i.e. $\tau \in \mathbb{H}$. If the torus defined by $(1, \tau_1)$ and $(1, \tau_2)$ are conformally equivalent, does that mean that

$\tau_1 = \tau_2$? Yes, because the arguments of τ_1 and τ_2 must be the same (otherwise angles would not be preserved by the conformal diffeomorphism), and then if they did not have the same magnitude, then the angle between 1 and $1+\tau_1$ would not be the same as the angle between 1 and $1+\tau_2$. Therefore, $\mathbb{H}$ is the parameter space for the set of all conformal types of tori.

We now work on the hyperbolic space, and we allow surfaces with boundary. Let $2g+n \geq 3$, and let $F_{g,n}$ be a fixed smooth, compact, oriented surface of genus g with n holes such that the boundary components are smooth closed curves. A **marked** Riemann surface of type (or **signature**) (g,n) is a pair (S,φ) such that S is a compact Riemann surface (assumed to be endowed with a metric of Gauss curvature -1) of type (g,n) and $\varphi : F_{g,n} \to S$ is a homeomorphism (called the **marking homeomorphism**). We say that $(S,\varphi) \sim (S',\varphi')$ (they are **equivalent markings**) if there exists an isometry $m : S \to S'$ such that φ' and $m \circ \varphi$ are isotopic. The **Teichmüller space $\mathcal{T}_{g,n}$ of type** (g,n) is defined to be

$$\mathcal{T}_{g,n} = \{[(S,\varphi)]\}.$$

It turns out that we could have used isotopy classes of such φ in place of φ, and also we could have restricted ourselves to diffeomorphisms φ. Note that if φ_1 and φ_2 are isotopic, then (S,φ_1) and (S,φ_2) are automatically marking equivalent, but the converse is false, because the m in the definition could be an isometry that is not isotopic to the identity. Also note that φ induces a hyperbolic structure on $F_{g,n}$.

There are many equivalent definitions. Let $\mathcal{H}$ be the set of all hyperbolic structures on $F_{g,n}$, and let Diff_0 be the group of diffeomorphisms of $F_{g,n}$ that are isotopic to the identity. Then

$$\mathcal{T}_{g,n} = \mathcal{H} \diagup \mathrm{Diff}_0.$$

Note that a hyperbolic structure is a subatlas of a (unique) conformal structure; by the uniformization theorem, every conformal structure contains a unique hyperbolic structure. Then

$$\mathcal{T}_{g,n} = \mathcal{C} \diagup \mathrm{Diff}_0,$$

where $\mathcal{C}$ is the set of smooth conformal structures in $F_{g,n}$ (smooth with respect to the smooth structure on $F_{g.n}$).

In the particular case $(g,n) = (0,3)$ (i.e. sphere with 3 holes), two maps $\varphi_1,\ \varphi_2 : F_{0,3} \to S$ are isotopic if and only if $\varphi_1^{-1} \circ \varphi_2$ fixes each boundary component of $F_{0,3}$. Such a (S,φ) is called a **marked Y-piece** or a **marked pair of pants**.

We construct the various Riemann surfaces $\{F(G,L,A)\}$ from Section 2, where $\omega = (L,A) = (\ell_1,...,\ell_{3g-3},\alpha_1,...,\alpha_{3g-3}) \in \mathbb{R}_{>0}^{3g-3} \times$

$\mathbb{R}^{3g-3} =: \mathcal{R}^{6g-6}$. These are called the Fenchel-Nielsen parameters. We will turn this set into a model of $\mathcal{T}_{g,0}$ by choosing suitable marking homeomorphisms. We will regard G as fixed and will label $F^\omega := F(G, L, A)$ for $\omega \in \mathcal{R}^{6g-6}$.

Lemma 5. *Let $\varphi_1, \varphi_2 : F \to S$ be two marking homeomorphisms. Then they are homotopic iff they are isotopic.*

Definition 6. *The set of all marked Riemann surfaces $S^\omega = (F^\omega, \varphi^\omega)$ based on the graph G is denoted $\mathcal{T}_G$.*

Theorem 7. *Let G be given. Then for every marked Riemann surface (S, φ), there exists a unique $S^\omega \in \mathcal{T}_G$ that is marking equivalent to (S, φ).*

Definition 8. *Given cubic graph G with $2g-2$ vertices, for every $S \in \mathcal{T}_g$, let $\omega(S) = \omega_G(S)$ denote the unique $\omega \in \mathbb{R}^{6g-6}$ such that S is marking equivalent to S_G^ω. The components are the **Fenchel-Nielsen coordinates** of S.*

Theorem 9. *If two cubic graphs G, G' with $2g-2$ vertices are given with coordinate maps ω_G and $\omega_{G'}$, then the transition function $\omega_G \circ \omega_{G'} : \mathbb{R}^{6g-6} \to \mathbb{R}^{6g-6}$ is a real analytic diffeomorphism.*

Definition 10. *The real analytic structure on $\mathcal{T}_g$ is given by the charts ω_G.*

Every homeomorphism $h : F \to F$ of the base surface to itself defines an action on $\mathcal{T}_g$ via $(S, \varphi) \mapsto (S, \varphi \circ h)$. Two such homeomorphisms that are isotopic define same action. Thus:

Definition 11. *The **mapping class group** Mod_g is the group of all equivalence classes of homeomorphisms $F \to F$ modulo isotopy.*

Note that orientation-reversing homeomorphisms are also allowed. Given $h \in \mathbf{M}_g$, let $m[h]$ denote the action on $\mathcal{T}_g$.

Definition 12. *The set*

$$\mathbf{M}_g : \{m[h] : h \in \mathrm{Mod}_g\}$$

*is called the **Teichmüller modular group**, and its elements are called **Teichmüller mappings**.*

Lemma 13. *For $g \geq 3$, $m : \mathrm{Mod}_g \to \mathbf{M}_g$ is an isomorphism. For $g = 2$, it has kernel $\mathbb{Z}_2$.*

Lemma 14. *Let $j : S \to S$ be an isometry. If j is isotopic to the identity, then it is the identity.*

Theorem 15. *$\mathcal{M}_g$ acts properly discontinuously on $\mathcal{T}_g$ by real analytic diffeomorphisms.*

Definition 16. *The quotient space*

$$\mathcal{T}_g \diagup \mathbf{M}_g$$

is the moduli space of hyperbolic Riemann surfaces of genus g.

Definition 17. *All of the above may be generalized to Riemann surfaces $S_{g,n}$ of genus g and n boundary components $\beta_1, ..., \beta_n$. Let $\mathcal{T}_{g,n}$ be the Teichmüller space of complete finite volume hyperbolic Riemann surfaces marked by $S_{g,n}$. Let $\mathrm{Mod}_{g,n}$ be the mapping class group, and let*

$$\mathcal{M}_{g,n} = \mathcal{T}_{g,n} \diagup \mathrm{Mod}_{g,n}$$

be the moduli space of hyperbolic Riemann surfaces of genus g with n ordered cusps.

The space $\mathcal{T}_{g,n}$ is a finite-dimensional complex manifold equipped with the Weil-Petersson symplectic form, to be described in the next section.

7. Symplectic geometry of moduli spaces of Riemann surfaces

We set notation similar to what we have already done. Let S be a fixed oriented smooth surface of negative Euler characteristic. A point in Teichmüller space $\mathcal{T}(S)$ is a complete hyperbolic surface X equipped with a diffeomorphism $f : S \to X$. The map f provides a **marking** on X by S. Two marked surfaces $f : S \to X$ and $g : S \to Y$ define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1} : Y \to X$ is isotopic to a conformal map. When ∂S is nonempty, consider hyperbolic Riemann surfaces homeomorphic to S with geodesic boundary components of fixed length. Let A be the set of components of ∂S, and let $b = (b_\alpha)_{\alpha \in A} \in \mathbb{R}_+^{|A|}$. A point $X \in \mathcal{T}(S, b)$ is a marked hyperbolic surface with geodesic boundary components such that for each boundary component $\beta \in A$, we have that the length of $f(\beta)$ is

$$\ell_\beta(X) = b_\beta.$$

Let $S_{g,n}$ be an oriented smooth connected surface of genus g with n boundary components $(\beta_1, ..., \beta_n)$. We let the Teichmüller space of hyperbolic structures on $S_{g,n}$ with geodesic boundary components of length $b_1, ..., b_n$ be

$$\mathcal{T}_{g,n}(b_1, ..., b_n) = \mathcal{T}(S_{g,n}, (b_1, ..., b_n)).$$

Let $\mathrm{Mod}(S)$ denote the mapping class group of S, i.e. the group of isotopy classes of orientation-preserving homeomorphisms from S to S leaving each boundary component β fixed (as a set). The mapping class group $\mathrm{Mod}_{g,n} = \mathrm{Mod}(S_{g,n})$ acts on $\mathcal{T}_{g,n}(b)$ by changing the marking. The quotient is

$$\begin{aligned} \mathcal{M}_{g,n}(b) &= \mathcal{M}(S_{g,n} : \ell_{\beta_i} = b_i \forall i) \\ &= \mathcal{T}_{g,n}(b) \big/ \mathrm{Mod}_{g,n}\ , \end{aligned}$$

and it is called the **moduli space of Riemann surfaces homeomorphic to $S_{g,n}$ with n boundary components of length** $\ell_{\beta_i} = b_i \forall i$. We allow boundary geodesics of length 0 (i.e. cusps), and so

$$\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0)\,; \mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0)\,.$$

For a disconnected surface S with components $\{S_i\}_{i=1}^k$ and boundaries $A_i = \partial S_i \subseteq \partial S$, we have

$$\mathcal{M}(S,b) = \prod_{i=1}^{k} \mathcal{M}(S_i, b_{A_i})\,,$$

where b_{A_i} is a vector of positive real numbers, one for each component of A_i.

Given $S_{g,n}$, fix a set $\mathcal{P} = \{\alpha_j\}_{j=1}^m$, where the α_j are disjoint simple closed curves which can be used to cut $S_{g,n}$ into pairs of pants. Note that $m = 3g-3+n$, this is the number of interior curves. For a marked hyperbolic surface $X \in \mathcal{T}_{g,n}(b)$, we let

$$FN(X) := (\ell(X), \tau(X)) = (\ell_{\alpha_1}(X), ..., \ell_{\alpha_m}(X), \tau_{\alpha_1}(X), ..., \tau_{\alpha_m}(X))\,,$$

with $\ell_{\alpha_j}(X)$ and $\tau_{\alpha_j}(X)$ being the lengths and twist parameters, respectively. Then

$$\mathcal{T}_{g,n}(b) \cong \mathbb{R}_+^m \times \mathbb{R}^m$$

by the FN map.

Theorem 18. *(Wolpert, 1981 CMH) The Weil-Petersson symplectic form is given by*

$$\omega_{wp} = \sum_{j=1}^{m} d\ell_{\alpha_j} \wedge d\tau_{\alpha_j}$$

This symplectic form of a Kähler metric that is not complete on the moduli space was introduced by A. Weil. Wolpert showed that it has a simple expression in terms of the Fenchel-Nielsen coordinates, and he showed that it extends as a closed form to the compactification $\overline{\mathcal{M}}_{g,n}$ and defines a cohomology class $[\omega] \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{R})$. In fact, $\left[\frac{\omega}{\pi^2}\right] \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, and by multiplying by an integer, this class would define a positive line bundle over $\overline{\mathcal{M}}_{g,n}$. This implies that $\overline{\mathcal{M}}_{g,n}$ is a projective algebraic variety. The Deligne-Mumford compactification is achieved by allowing $\ell_\gamma = 0$ for simple closed geodesics inside $S_{g,n}$. When the length vector $\ell(X) = 0$, there is a natural complex structure on $\mathcal{T}_{g,n}$, and this symplectic form is actually the Kähler form of a Kähler metric. When b is nonzero, there is no natural complex structure on $\mathcal{M}_{g,n}(b)$, but as mentioned before, the Fenchel-Nielsen coordinates give it a real analytic structure.

Given a simple closed geodesic α on $X \in \mathcal{T}_{g,n}(b)$ and $t \in \mathbb{R}$, we can deform the complex structure of X by cutting along α and reglue back after twisting t units to the right. We denote the new surface by $\mathrm{tw}_\alpha^t(X)$. This one-parameter family is a continuous path in Teichmüller space. For the particular case $t = \ell_\alpha(X)$, $\mathrm{tw}_\alpha^t(X) = \phi_\alpha(X)$, where $\phi_\alpha \in \mathrm{Mod}(S_{g,n})$ is a right **Dehn twist** about α, and thus for this t we are back to the same class in $\mathcal{M}_{g,n}(b)$.

8. Digression: symplectic reduction

A **symplectic manifold** (M, ω) is a smooth manifold M of dimension $2n$ along with a nondegenerate closed 2-form ω, called the symplectic form. It is closed in that $d\omega = 0$, and it is nondegenerate in that the symplectic volume form $\frac{1}{n!}\omega^n = \frac{1}{n!}\omega \wedge \ldots \wedge \omega$ is never zero on M. A **symplectomorphism** $\phi : M \to M'$ is a diffeomorphism between symplectic manifolds (M, ω) and (M', ω') such that $\omega = \phi^*\omega'$. The Darboux Theorem states that all symplectic manifolds are locally symplectomorphic to $(\mathbb{R}^{2n}, dp^1 \wedge dq^1 + \ldots + dp^n \wedge dq^n)$ (**canonical coordinates**). This is the model of a configuration space of a particle in n-dimensions; q = position, p = momentum.

Given a one-form α on M, we say that the vector field $\alpha^\#$ is the **symplectic dual** of α if $\alpha(X) = \omega\left(\alpha^\#, X\right)$ for all vector fields X. Further, a function H called a **Hamiltonian** function is associated to a **Hamiltonian vector field** ξ_H if

$$dH(X) = \omega(\xi_H, X)$$

for all vector fields X. The vector field ξ_H is also called the **symplectic gradient** of H. The prototypical example of this is $H = \frac{1}{2}\sum p_j^2 + \frac{1}{2}\sum q_k^2$ (total energy of a particle of mass 1 attached to a spring with spring constant 1). Then

$$dH = \sum (p_j dp_j + q_j dq_j) = r dr$$

(with r the polar coordinate). Observe that if

$$\xi_H = \partial_\theta = \sum \left(q_j \partial_{p_j} - p_j \partial_{q_j}\right)$$

Then

$$\begin{aligned}\omega\left(\xi_H,\cdot\right) &= \sum_{j,k}\left(dp_j\wedge dq_j\right)\left(q_k\partial_{p_k}-p_k\partial_{q_k}\right)\\ &= \sum_j\left(q_jdq_j+p_jdp_j\right)=dH.\end{aligned}$$

Note that in this case, ξ_H is the vector field generated by the rotational symmetries around the two dimensional subspaces. By Noether's Theorem, every symmetry that leaves the symplectic form invariant (like rotation around the two plane) gives rise to a **first integral of motion** — ie an invariant function H. Note that in all cases the flow of the Hamiltonian vector field preserves the Hamiltonian function and also is a family of symplectomorphisms.

Now, suppose that a whole group G acts on (M,ω) by symplectomorphisms. We define the **moment map**

$$\mu : M\to\mathfrak{g}^*$$

from M to the dual of the Lie algebra $\mathfrak{g}$ of G by the formula

$$d\mu\left(Y\right)\left(X\right)=\omega\left(\overline{X},Y\right)$$

for all vector fields Y and all $X\in\mathfrak{g}$. The vector field $\overline{X}$ on M is the vector field induced by X, ie

$$\overline{X}_p=\left.\frac{d}{dt}\right|_{t=0}\exp^G\left(tX\right)\cdot p$$

for $p\in M$. What this means is that X represents an infinitessimal symmetry of (M,ω), which by Noether's theorem generates a conserved quantity, the function $p\mapsto\mu\left(p\right)\left(X\right)$, which is a Hamiltonian for $\overline{X}$. For example, if $\overline{X}$ comes from a rotation, $\mu\left(\cdot\right)\left(X\right)$ is the angular momentum from that rotation. A really simple example for this is in $\mathbb{R}^2$, note that

$$d\left(\frac{r^2}{2}\right)=i\left(\partial_\theta\right)\left(dy\wedge dx\right),$$

where $r^2=x^2+y^2$ and $\partial_\theta=xdy-ydx$. And note that the angular momentum of a unit point mass rotating from (x,y) around the circle centered at the origin at velocity ∂_θ is $vr=x^2+y^2$. (OK, make it mass $\frac{1}{2}$ so that the formula matches.)

For another example, let N be a smooth manifolds, and let $M=T^*N$ be the cotangent bundle. Let τ be the tautological one-form on M. (That is, $\pi : T^*N\to N$ means that $\pi_* : T\left(T^*N\right)\to TN$, so for $\alpha_x\in T_x^*N$, $\xi_{\alpha_x}\in T_{\alpha_x}\left(T^*N\right)$, we define $\tau\in\Omega^1\left(T^*N\right)$ by $\tau_{\alpha_x}\left(\xi_{\alpha_x}\right)=\alpha_x\left(\pi_*\left(\xi_{\alpha_x}\right)\right)\in\mathbb{R}$.) Then we let $\omega=d\tau\in\Omega^2\left(T^*N\right)$. In canonical coordinates $\tau = \sum p^i dq_i$ and $d\tau = \sum dp_i \wedge dq_i$. Suppose now that the Lie group G acts on the base manifold N. Then there is an induced action on T^*N given by $g \cdot \alpha_x := (g^{-1})^* (\alpha_x) \in T^*_{gx}N$. This action is Hamiltonian with the moment map defined for $X \in \mathfrak{g}$ by

$$\mu(\bullet)(X) = -i\left(\overline{X}\right)\tau.$$

Now, if $a \in \mathfrak{g}^*$ is a regular value of μ, we note that

$$\mu^{-1}(a)$$

is a submanifold of codimension $\dim G$, and G acts on $\mu^{-1}(a)$ (because the Hamiltonian is preserved under the Hamiltonian flow). We define

$$M_a = \mu^{-1}(a) \diagup G$$

to be the symplectic quotient of M at a. Here, we have reduced the number of variables by using the symmetry of the system. Note that the symplectic form descends to a symplectic form on M_a – we have reduced the dimension by two times the dimension of G, and so we have a nondegenerate two-form that is closed on the quotient.

If 0 is a regular value of the moment map, the **coisotropic embedding theorem** states that there is a neighborhood of the submanifold $\mu^{-1}(0)$ on which the symplectic form is given in a standard form. This is a generalization of Darboux's theorem. In this case, when a is close to zero, M_a is diffeomorphic to M_0. If $G = T^n = (S^1)^n$, the action of G on the level set $\mu^{-1}(a)$ gives rise to n circle bundles $\mathcal{C}_1, ..., \mathcal{C}_n$ defined over M_a. Fix a connection α of this T^n bundle on $\mu^{-1}(0)$. The curvature Ω of α satisfies $c_1(\mathcal{C}) = c_1(\mathcal{C}_1 \oplus ... \oplus \mathcal{C}_n) = [\Omega]$.

Theorem 19. *(Normal Form Theorem, see Guillemin) The space (M_a, ω_a) is symplectomorphic to $(M_0, \omega_0 + a\Omega)$, where Ω is the curvature form of the connection α. As cohomology classes,*

$$[\omega_a] = [\omega] + \sum_{j=1}^{n} a_j \cdot [\phi_j],$$

where $\phi_j = c_1(\mathcal{C}_j)$ denotes the first Chern class.

We now integrate ω_a^m over M_a to get the following.

Corollary 20. *Let 0 be a regular value of the proper moment map $\mu : M \to \mathbb{R}^n$ of the Hamiltonian action of T^n on M. Then for sufficiently small $\varepsilon > 0$, $a \in \mathbb{R}^n_+$ with $|a| \leq \varepsilon$, the volume of $M_\alpha = \mu^{-1}(a) \diagup T^n$ is a polynomial in $a_1, ..., a_n$ of degree $m = \dim(M_a)/2$ given by*

$$\sum_{|\beta| \leq m} C(\beta) \cdot a^\beta,$$

where

$$C(\beta) = \frac{1}{\beta!\,(m - |\beta|)!}\int_{M_0} \phi_1^{\beta_1}...\phi_n^{\beta_n} \cdot \omega^{m-|\beta|}$$

9. Back to symplectic geometry of the moduli space

For any closed geodesic γ on a hyperbolic surface X, there is a collar neighborhood of width

$$\operatorname{arcsin h}\left(\frac{1}{\sinh\left(\ell_\gamma(X)/2\right)}\right)$$

that is an embedded annulus. Moreover, two simple closed geodesics are disjoint iff their collars are disjoint. Therefore, for each boundary component β_j of $X \in \mathcal{T}_{g,n}(b)$, there is a curve $\widetilde{\beta_j}$ of constant curvature of length close to $\ell_{\beta_j}(X)$ inside the collar neighborhood of β_j. In the case where the length of the boundary component β_j is taken to zero, $\widetilde{\beta_j}$ tends to the horocycle of length $\frac{1}{4}$ around the puncture. When β_j does not have length 0, there is a canonical bijection between points of $\widetilde{\beta_j}$ and those of β_j.

The orientation of $S_{g,n}$ induces an orientation on its boundary components. Let $\gamma_j : [0, b_j] \to \beta_j$ be arclength parametrizations. For $t \in [0,1]$, we define $\xi^t : \beta_j \to \beta_j$ by

$$\xi^t(\gamma_j(s)) = \gamma_j(s + tb_j).$$

So since $\xi^{t+1} = \xi^t$, this is a circle action.

Next, let $\widetilde{\beta_j}$ a curve parallel to the boundary component as above. The advantage to using $\widetilde{\beta_j}$ instead of β_j is that it has positive length even when β_j has length 0. Note for $i \neq j$, $\widetilde{\beta_i}$ is disjoint from $\widetilde{\beta_j}$. For fixed $b = (b_1, ..., b_n)$, define

$$\mathcal{S}_i(\mathcal{T}_{g,n}(b)) = \left\{(X,p) : p \in \widetilde{\beta_i},\ X \in \mathcal{T}_{g,n}(b)\right\} \to \mathcal{T}_{g,n}(b),$$

a circle bundle. Quotienting out by the natural action of $\mathrm{Mod}_{g,n}$, we still get a circle bundle

$$\mathcal{S}_i(\mathcal{M}_{g,n}(b)) \to \mathcal{M}_{g,n}(b)$$

in the orbifold sense since the stabilizers are finite. Also, the circle bundle can be extended to $\overline{\mathcal{M}_{g,n}}(b)$:

$$\mathcal{S}_i\left(\overline{\mathcal{M}_{g,n}}(b)\right) \to \overline{\mathcal{M}_{g,n}}(b)$$

Since this is a circle bundle $\mathcal{S}_i$, its (first) Chern class $[c_1(\mathcal{S}_i)] \in H^2\left(\overline{\mathcal{M}_{g,n}}, \mathbb{Q}\right)$ (note, it's an orbifold). We need to relate this to the tautological class $\psi_i = c_1(\mathcal{L}_i)$, where the line bundle $\mathcal{L}_i$ is the cotangent space over the marked point x_i. Recall that via the uniformization theorem there is a unique compact complex curve C and points $p_1, ..., p_n$ on C such that X is conformally equivalent to $C\backslash\{p_1, ..., p_n\}$. Also, each cusp neighborhood of X is equivalent to $\Delta \setminus \{0\} \subset \mathbb{C}$. Considering the parallel curve $\widetilde{\beta}_j$ around the puncture p_j, each element of the tangent space of X at p_j corresponds to a point of $\widetilde{\beta}_j$. However, the orientation of $\widetilde{\beta}_j$ is the negative of the one induced by the tangent vectors at p_i. On the other hand, as $\mathcal{L}_j$ is a complex line bundle, the orientation on the cotangent space is the opposite of that of the tangent space, so there is an orientation-reversing isomorphism between the circle bundle $\mathcal{S}_j$ and $\mathcal{L}_j$. Thus,

$$[c_1(\mathcal{S}_i)] = \psi_i = [c_1(\mathcal{L}_i)] \in H^2\left(\overline{\mathcal{M}_{g,n}}, \mathbb{Q}\right).$$

Next, let the moduli space of bordered Riemann surfaces with marked points (without fixing the lengths of the boundary components) be defined by

$$\widehat{\mathcal{M}_{g,n}} = \left\{(X, p_1, ..., p_n) : X \in \overline{\mathcal{M}_{g,n}}(b_1, ..., b_n), \forall j, p_j \in \widetilde{\beta}_j, b_j > 0\right\}.$$

Define the map $\ell : \widehat{\mathcal{M}_{g,n}} \to \mathbb{R}_+^n$ by

$$\ell(X, p_1, ..., p_n) = (\ell_{\beta_1}(X), ..., \ell_{\beta_n}(X)).$$

There is a natural action of $T^n = (S^1)^n$ on the space $\widehat{\mathcal{M}_{g,n}}$ as follows. For each j, S^1 acts by moving p_j along $\widetilde{\beta}_j$, i.e.

$$\xi_j^t(X, p_1, ..., p_n) = \left(X, p_1, ..., \xi^t(p_j), ..., p_n\right).$$

It turns out that this T^n action is the Hamiltonian flow of the function "$\ell^2/2$" with respect to the symplectic form on $\widehat{\mathcal{M}_{g,n}}$ induced by the Weil-Petersson form.

Theorem 21. *The orbifold $\widehat{\mathcal{M}_{g,n}}$ has a natural T^n-invariant symplectic structure such that*

(1) *the map*

$$\ell^2/2 = \left(\ell_{\beta_1}(X)^2/2, ..., \ell_{\beta_n}(X)^2/2\right)$$

is the moment map for the action of T^n on $\widehat{\mathcal{M}_{g,n}}$, and

(2) *the canonical map*

$$s : \ell^{-1}(b_1, ..., b_n) \diagup T^n \to \overline{\mathcal{M}_{g,n}}(b_1, ..., b_n)$$

is a symplectomorphism.

We remark that this extension of the symplectic form to $\overline{\mathcal{M}_{g,n}}(0, ..., 0)$ is the Weil-Pederson symplectic form.

Theorem 22. *The coefficients of the volume polynomial*

$$\mathrm{VOL}\left(\mathcal{M}_{g,n}\left(b_1,...,b_n\right)\right)=\sum_{|\alpha|\leq 3g-3+n} C_g\left(\alpha\right) b^{2\alpha}$$

are given by

$$C_g\left(\alpha_1,...,\alpha_n\right)=\frac{2^{m(g,n)|\alpha|}}{2^{|\alpha|}\,|\alpha|!\left(3g-3+n-|\alpha|\right)!}\int_{\overline{\mathcal{M}_{g,n}}}\psi_1^{\alpha_1}...\psi_n^{\alpha_n}\cdot\omega^{3g-3+n-|\alpha|},$$

where $\psi_j=c_1\left(\mathcal{L}_j\right)$, ω *is the Weil-Petersson symplectic form,* $m\left(g,n\right)=\delta\left(g-1\right)\delta\left(n-1\right)$.

Remark 23. *The coefficient* $C_g\left(\alpha\right)$ *is positive and lies in* $\pi^{6g-6+2n-2|\alpha|}\mathbb{Q}$.

Remark 24. *By a result of Wolpert,* $\kappa_1=\frac{[\omega]}{2\pi}$ *is the first Mumford tautological class on* $\overline{\mathcal{M}_{g,n}}$.

10. Recursive formula for Weil-Pedersson volumes

To get a recursive formula for the Weil-Pedersson volume, we need the Generalized McShane Identity, as follows.

Theorem 25. *(Generalized McShane identity for bordered surfaces) For any* $X\in\mathcal{T}_{g,n}\left(b_1,...,b_n\right)$ *with* $3g-3+n>0$, *we have*

$$\sum_{(\alpha_1,\alpha_2)}\mathcal{D}\left(b_1,\ell_{\alpha_1}\left(X\right),\ell_{\alpha_2}\left(X\right)\right)+\sum_{i=2}^{n}\sum_{\gamma}\mathcal{R}\left(b_1,b_i,\ell_\gamma\left(X\right)\right)=b_1\ ,$$

where the first sum is over all unordered pairs of simple closed geodesics (α_1,α_2) *bounding a pair of pants with boundary component* β_1, *and the second sum is over simple closed geodesics* γ *bounding a pair of pants with* β_1 *and* β_i. *The two functions are* $\mathcal{D},\mathcal{R}:\mathbb{R}^3\to\mathbb{R}_+$ *with*

$$\begin{aligned}\mathcal{D}\left(x,y,z\right)&=2\log\left(\frac{\exp\left(\frac{x}{2}\right)+\exp\left(\frac{y+x}{2}\right)}{\exp\left(-\frac{x}{2}\right)+\exp\left(\frac{y+x}{2}\right)}\right),\\ \mathcal{R}\left(x,y,z\right)&=x-\log\left(\frac{\cosh\left(\frac{y}{2}\right)+\cosh\left(\frac{x+z}{2}\right)}{\cosh\left(\frac{y}{2}\right)+\cosh\left(\frac{x-z}{2}\right)}\right).\end{aligned}$$

Remark 26. *In a pair of pants with boundary components* β_1, β_2, β_3 *of lengths* x_1, x_2, x_3, *there are three interesting geodesics that intersect* β_1 *at right angles, at the six intersection points* y_1,w_1,z_1,z_2,w_2,y_2 *(in order). There is a unique geodesic from* y_1 *to* y_2 *that spins around the pant leg with component* β_3, *and there is a unique geodesic that spins around the pant leg with component* β_2 *that goes from* z_1 *to* z_2, *and* *finally there is a unique geodesic that goes down the "crotch" of the pair of pants from* w_1 *to* w_2*. Then*

$$\begin{aligned} \mathcal{R}(x_1,x_2,x_3) &= d(y_1,y_2) \\ x_1-\mathcal{R}(x_1,x_2,x_3) &= \textit{projection of } \beta_3 \textit{ to } \beta_1 \textit{ in univ. cover} \\ \mathcal{D}(x_1,x_2,x_3) &= d(y_1,z_1)+d(y_2,z_2) \\ &= 2d(\textit{proj } \beta_2 \textit{ to } \beta_1, \textit{proj } \beta_3 \textit{ to } \beta_1). \end{aligned}$$

From this we see that $\mathcal{D}$ *is symmetric wrt* $x_2 \leftrightarrow x_3$*, and*

$$\mathcal{R}(x_1,x_2,x_3)+\mathcal{R}(x_1,x_3,x_2)=x_1+\mathcal{D}(x_1,x_2,x_3).$$

All of the formulas in the theorem above can be derived using hyperbolic trigonometry.

Note that if

$$H(x,y)=\frac{1}{1+\exp\left(\frac{x+y}{2}\right)}+\frac{1}{1+\exp\left(\frac{x-y}{2}\right)},$$

Then

$$\begin{aligned} \frac{\partial}{\partial x}\mathcal{D}(x,y,z) &= H(y+z,x)\ , \\ \frac{\partial}{\partial x}\mathcal{R}(x,y,z) &= \frac{H(z,x+y)+H(z,x-y)}{2}. \end{aligned}$$

The first step is to calculate $V_{1,1}(b)$. By the McShane Identity, for any $X\in\mathcal{T}(S_{1,1},b)$, we have

$$\sum_\gamma \mathcal{D}(b,\ell_\gamma(X),\ell_\gamma(X))=b,$$

where the sum is over all nonperipheral simple closed curves on $S_{1,1}$. It is then possible to integrate this over $\mathcal{M}_{1,1}(b)$. To do this, observe that there is a single simple closed geodesic curve of length $x>0$ on any such surface $S_{1,1}$ that when cut produces a hyperbolic pair of pants. The parameters of the moduli space are the length x and also the twisting (gluing) parameter t, and the symplectic (volume) form is $dt\wedge dx$. Note that $0\le t\le x$ in the moduli space, and so we integrate the equation to get

$$\begin{aligned} \int_0^\infty\int_0^x \mathcal{D}(b,x,x)\ dt\wedge dx &= b\cdot V_{1,1}(b)\,, \text{ or} \\ \int_0^\infty x\mathcal{D}(b,x,x)\ dx &= b\cdot V_{1,1}(b) \end{aligned}$$

From the explicit equations above,

$$\frac{\partial}{\partial b}\mathcal{D}(b,x,x)=\frac{1}{1+\exp\left(x-\frac{b}{2}\right)}+\frac{1}{1+\exp\left(x+\frac{b}{2}\right)}.$$

So we have

$$\frac{\partial}{\partial b}(b\cdot V_{1,1}(b))=\int_0^\infty x\left(\frac{1}{1+\exp\left(x-\frac{b}{2}\right)}+\frac{1}{1+\exp\left(x+\frac{b}{2}\right)}\right)\,dx.$$

We set $y_1=x+\frac{b}{2}$, $y_2=x-\frac{b}{2}$, we get

$$\begin{aligned}&\int_0^\infty x\left(\frac{1}{1+\exp\left(x-\frac{b}{2}\right)}+\frac{1}{1+\exp\left(x+\frac{b}{2}\right)}\right)\,dx\\ =\;&\int_{b/2}^\infty\frac{y_1-\frac{b}{2}}{1+\exp(y_1)}dy_1+\int_{-b/2}\frac{y_2+\frac{b}{2}}{1+\exp(y_2)}dy_2=...\\ =\;&\frac{\pi^2}{6}+\frac{b^2}{8}.\end{aligned}$$

using the fact that $\frac{1}{1+e^y}+\frac{1}{1+e^{-y}}=1$. As a result, since the above is $\frac{\partial}{\partial b}(b\cdot V_{1,1}(b))$, we get

$$V_{1,1}(b)=\frac{b^2}{24}+\frac{\pi^2}{6}.$$

Also, the absolutely most trivial case is when $g=0$ and $n=3$. This is a single pair of pants yielding exactly one point in the moduli space, and by definition

$$V_{0,3}(b_1,b_2,b_3)=1.$$

Next, let $L=\{L_1,...,L_n\}$ be a set of positive numbers, and let

$$V_{g,n}(L)=V_{g,n}(L_1,...,L_n).$$

We now state the recursive formula of Mirzakhani.

Theorem 27. *The following recursive formula for $V_{g,n}(L)$ holds, and using this recursion, all of these polynomials can be computed.*

- For any $L_1,L_2,L_3\geq 0$, set
$$\begin{aligned}V_{0,3}(L_1,L_2,L_3)&=1\\ V_{1,1}(L_1)&=\frac{L_1^2}{24}+\frac{\pi^2}{6}.\end{aligned}$$
- For $L=(L_1,...,L_n)$, let $\widehat{L}=(L_2,...,L_n)$. For $(g,n)\neq(1,1),(0,3)$, the volume satisfies

$$\frac{\partial}{\partial L_1}(L_1V_{g,n}(L))=\mathcal{A}^{con}_{g,n}\left(L_1,\widehat{L}\right)+\mathcal{A}^{dcon}_{g,n}\left(L_1,\widehat{L}\right)+\mathcal{B}_{g,n}\left(L_1,\widehat{L}\right),$$

where

- $\mathcal{A}^{con}_{g,n}\left(L_1,\widehat{L}\right)=\frac{1}{2}\left(\int_0^\infty\int_0^\infty xy\widehat{\mathcal{A}}^{con}_{g,n}\left(x,y,L_1,\widehat{L}\right)\ dx\ dy\right)$
- $\mathcal{A}^{dcon}_{g,n}\left(L_1,\widehat{L}\right)=\frac{1}{2}\left(\int_0^\infty\int_0^\infty xy\widehat{\mathcal{A}}^{dcon}_{g,n}\left(x,y,L_1,\widehat{L}\right)\ dx\ dy\right)$
- $\mathcal{B}_{g,n}\left(L_1,\widehat{L}\right)=\int_0^\infty x\widehat{\mathcal{B}}_{g,n}\left(x,L_1,\widehat{L}\right)\ dx$

where

- $\widehat{\mathcal{A}}^{con}_{g,n}\left(x,y,L_1,\widehat{L}\right)=\frac{1}{2^{m(g-1,n+1)}}V_{g-1,n+1}\left(x,y,\widehat{L}\right)H\left(x+y,L_1\right)$
- $\widehat{\mathcal{A}}^{dcon}_{g,n}\left(x,y,L_1,\widehat{L}\right)=\sum_{(g_1,I_1,g_2,I_2)}\frac{V_{g_1,n_1+1}\left(x,L_{I_1}\right)}{2^{m(g_1,n_1+1)}}\frac{V_{g_2,n_2+1}\left(x,L_{I_2}\right)}{2^{m(g_2,n_2+1)}}$,
  where the sum is over all $I_1,I_2\subset\{2,...,n\}$ and $0\leq g_1,g_2\leq g$ such that
  - $I_1\sqcup I_2$ is a partition of $\{2,...,n\}$
  - $2\leq 2g_j+|I_j|\,,g_1+g_2=g$
- $\widehat{\mathcal{B}}_{g,n}\left(x,L_1,\widehat{L}\right)$
  $=\frac{1}{2^{m(g,n-1)}}\sum_{j=2}^n\frac{1}{2}\left(H\left(x,L_1+L_j\right)+H\left(x,L_1-L_j\right)\right)V_{g,n-1}\left(x,L_2,...,\widehat{L_j},...,L_n\right)$

The idea of proof of this theorem is the following. We cut out a pair of pants, one of whose boundary components is β_1, from our surface $S_{g,n}\left(L\right)$, and after removing this, there are three possibilities. The first possibility is that we other two pants legs are interior circles in the surface, and removing the pair of pants results in no additional new components of the Riemann surface, so the genus is reduced by one, and the number of boundary components goes up by one. The next possibility is that the other two pants legs are interior circles in the surface, and removing the pair of pants results in two separated Riemann surfaces with boundary, where we have divided up the boundary components (including the two new ones) and the genera among the two pieces. The third possibility is that β_1 is one pant leg boundary, β_j is another pant leg boundary, and the last pant leg boundary is an interior circle. In this case, the number of boundary components goes down by 1, and the genus stays the same. These three possibilities correspond exactly to the three terms $\mathcal{A}^{con}_{g,n}\left(L_1,\widehat{L}\right)$, $\mathcal{A}^{dcon}_{g,n}\left(L_1,\widehat{L}\right)$, $\mathcal{B}_{g,n}\left(L_1,\widehat{L}\right)$ in the recursive formula.

Using this formula, one may calculate $V_{g,n}\left(L\right)$. Previously, only formulas for $V_{1,1}\left(L\right)$ and $V_{0,n}\left(L\right)$ were known. Here is a sampling of these volume polynomials:

g	n	$V_{g,n}(L)$
0	3	1
1	1	$\frac{1}{24}\left(L^2+4\pi^2\right)$
0	4	$\frac{1}{2}\left(4\pi^2+\sum L_j^2\right)$
1	2	$\frac{1}{192}\left(4\pi^2+L_1^2+L_2^2\right)\left(12\pi^2+L_1^2+L_2^2\right)$
2	1	$\frac{1}{2211840}\left(4\pi^2+L^2\right)\left(12\pi^2+L^2\right)\left(6960\pi^4+384\pi^2L^2+5L^4\right)$