FOLIATIONS, METRICS, AND MEAN CURVATURE

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Abstract: Given a Riemannian manifold, we suppose that we are given a foliation on the manifold, i.e. a layering of immersed submanifolds. We will discuss the mean curvature vector fields and dual one-forms associated to this structure and explain their meaning. We show how to modify one metric into any other, and we quantify how the modification affects the mean curvature. Part of this talk contains joint work with Igor Prokhorenkov and Marco Radeschi.

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1. What is mean curvature?

Let (M, g) be a Riemannian manifold of dimension n. Many of the computations we will use are local, and in that case there is no reason to assume M is closed (compact and without boundary). When we start talking about global quantities like cohomology, we will probably restrict to the closed manifold case.

Let N be a submanifold of M of dimension p. We do not necessarily assume it is embedded, so for example, it could be the set $\left\{(x, y) = \mathbb{R}^2 \big/ \mathbb{Z}^2 : x = \sqrt{2}y\right\}$, which forms a dense, immersed submanifold in the torus $T^2 = \mathbb{R}^2 \big/ \mathbb{Z}^2$.

Let $\{f_\alpha\}_{1\leq\alpha\leq p}$ be a local orthonormal frame of $N \subseteq M$. Then we can take the covariant derivative $\nabla_{f_\alpha} f_\alpha$ in direction f_α and compute its component $(\nabla_{f_\alpha} f_\alpha)^\perp$ orthogonal to the submanifold N. The mean curvature vector field H of N is defined to be the vector field on M defined locally along N that is given by the formula

$$H = \sum_{\alpha=1}^{p} (\nabla_{f_\alpha} f_\alpha)^\perp .$$

We could check that the definition does not depend on the choice of orthonormal frame. If the local vector fields is extended in any way to be defined in an M-neighborhood of the point of N, we would get the same result when restricted to N. The submanifold N is called **minimal** if $H = 0$ at all points.

The mean curvature is the force vector of surface tension, up to a constant. Note that by rescaling the metric by a constant c^2 (shrinking N and M), the mean curvature vector field gets multiplied by $\frac{1}{c^2}$.

The mean curvature form κ is the one-form along N defined as $\kappa\left(v_x\right) = \left\langle v_x, H_x \right\rangle$ for $x \in N$, $v_x \in T_x M$; i.e. we say $\kappa = H^{\flat}$. Again, this can be extended in an M-neighborhood. This form contains the same information as H but does not change when the metric is rescaled. It is still true that $\kappa = 0$ if and only if the submanifold N is minimal.

The form κ can be computed using **Rummler's formula:**

$$d\chi = -\kappa \wedge \chi + \varphi,$$

where χ is the volume form along the submanifold (extended in a neighborhood of N, also called **characteristic form**), and φ is a $(p+1)$-form on a neighborhood of N such that $v_1 \lrcorner v_2 \lrcorner ... v_p \lrcorner \varphi = 0$ on N whenever all of the vectors $v_1, ..., v_p$ are tangent to N. For a k-form α, if v is a vector at a point x, then $v \lrcorner \alpha$ is the element of $\wedge^{k-1} T_x^* M$ defined by $v \lrcorner \alpha = \alpha(v, \cdot, ..., \cdot)$. So the condition on φ says roughly that φ has at most $p-1$ "directions" along N. The pieces of this formula may be identified as the $(1, p)$ and $(2, p-1)$ components of $d\chi$, where the first index indicates the number of components normal to the submanifold direction and the second index indicates the directions in the submanifold direction.

Rummler's formula is most often used in the context of a foliation (layering of submanifolds like N so that locally the manifold if diffeomorphic to a product of manifolds). In this case, we may locally choose an adapted orthonormal frame $f_1, ..., f_p, e_1, ..., e_q$ for the tangent bundle, where $\{f_1, ..., f_p\}$ spans the tangent space $T\mathcal{F}$ to the "leaves" (local submanifolds) and $\{e_1, ..., e_q\}$ spans the normal bundle $N\mathcal{F}$ to the leaves. Then the duals of these vector fields $f^1, ..., f^p, e^1, ..., e^q$ form a local adapted orthonormal basis for the cotangent bundle $T^*M = T^*\mathcal{F} \oplus N^*\mathcal{F}$. This could also certainly be done for any distribution of rank p, i.e. a subbundle of the tangent bundle that it is not necessarily integrable. With this adapted coframe, the characteristic form χ satisfies

$$\chi = f^1 \wedge f^2 \wedge ... \wedge f^p.$$

Aside: If we use the Levi-Civita connection to calculate covariant derivatives of elements of an orthonormal frame $\{v_j\}$ and coframe $\{v^j\}$, we obtain the formula

$$\nabla_{v_j} v_k = \sum_s \omega_{jk}^s v_s,$$

for some functions ω_{jk}^s dependent on the metric, with symmetry $\omega_{jk}^s = -\omega_{js}^k$. Using properties of tensor derivations, we obtain corresponding derivatives for coframe elements:

$$\nabla_{v_j} v^k = -\sum_s \omega_{js}^k v^s = \sum_s \omega_{jk}^s v^s.$$

We note also that this formula can be used to calculate the differentials of these one-forms:

$$d\left(v^j\right) = \sum_i v^i \wedge \nabla_{v_i} v^j = -\sum_{i,k} \omega_{ik}^j v^i \wedge v^k.$$

Now, suppose a distribution of rank p has a local adapted orthonormal frame $f_1, ..., f_p, e_1, ..., e_q$ with corresponding adapted coframe $f^1, ..., f^p, e^1, ..e^q$. We will use Greek indices to indicate the f. indices, and Roman indices to indicate the e. indices. Then we have

$$\begin{aligned}
\chi &= f^1 \wedge f^2 \wedge ... \wedge f^p \\
d\chi &= \sum_\alpha (-1)^{\alpha+1} f^1 \wedge ... \wedge df^\alpha \wedge ... \wedge f^p \\
&= \sum_\alpha (-1)^{\alpha+1} f^1 \wedge ... \wedge \left(\sum_{k,\beta} \left(-\omega^\alpha_{k\beta} e^k \wedge f^\beta - \omega^\alpha_{\beta k} f^\beta \wedge e^k\right) - \sum_{r,s} \omega^\alpha_{rs} e^r \wedge e^s \right) \\
&\qquad \wedge ... \wedge f^p \\
&= \sum_{\alpha,k} \left(\omega^\alpha_{\alpha k} - \omega^\alpha_{k\alpha}\right) e^k \wedge \chi + \left(\sum_{\alpha,r,s} (-1)^\alpha \omega^\alpha_{rs} e^r \wedge e^s \wedge f^1 \wedge ... \wedge \widehat{f^\alpha} \wedge ... \wedge f^p \right) \\
&= -\kappa \wedge \chi + \varphi,
\end{aligned}$$

where the two pieces of the formula have the desired properties, with κ of type $(1,0)$ and φ of type $(2, p-1)$.

Furthermore

$$\begin{aligned}
\kappa &= \sum_{\alpha,k} \left(\omega^{\alpha}_{k\alpha} - \omega^{\alpha}_{\alpha k}\right) e^k = \sum_{\alpha,k} \left(0 - \omega^{\alpha}_{\alpha k}\right) e^k \\
&= \sum_{\alpha,k} \omega^{k}_{\alpha\alpha} e^k = \left(\sum_{\alpha,k} \omega^{k}_{\alpha\alpha} e_k\right)^{\flat} \\
&= \left(\sum_{\alpha} \left(\nabla_{f_\alpha} f_\alpha\right)^{\perp}\right)^{\flat} = H^{\flat}.
\end{aligned}$$

We note also that the form φ is zero if and only if the normal bundle to the distribution is involutive (i.e. forms the tangent space to a foliation).

The following are other equivalent formulas for mean curvature:

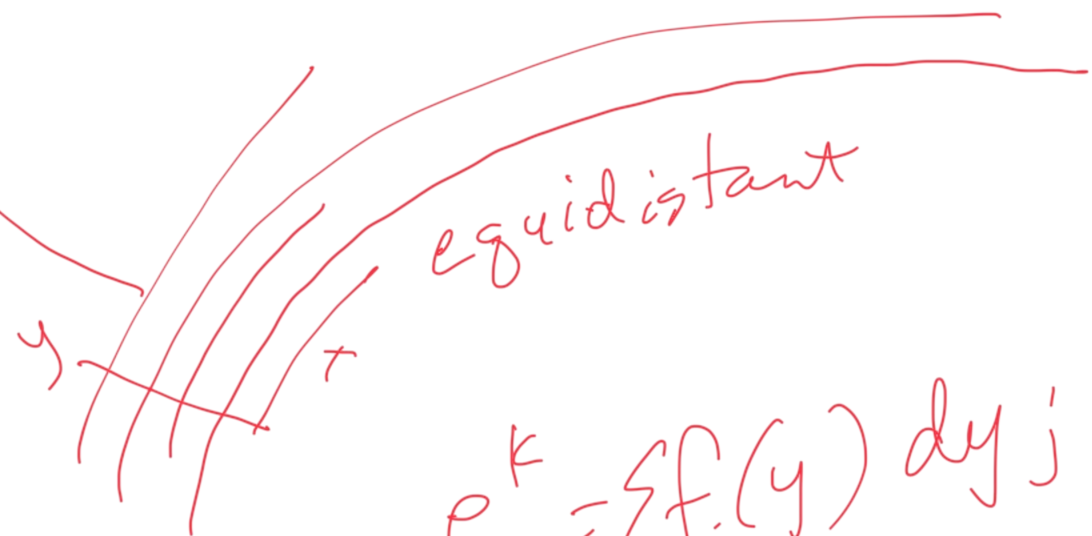
$$\begin{aligned}
\kappa &= (-1)^{p+1}\,\chi\lrcorner d\chi, \\
\kappa &= \sum_{\alpha,k}\left(f_\alpha\lrcorner df^\alpha, e^k\right)e^k = \sum_{\alpha,k}\left(e_k\lrcorner f_\alpha\lrcorner df^\alpha\right)e^k, \\
\kappa &= \sum_{\alpha} f_\alpha\lrcorner d_{1,0}f^\alpha, \\
\kappa &= -\sum_{\alpha,k}(f_\alpha,[f_\alpha,e_k])e^k.
\end{aligned}$$

Local conditions on the distribution:

(1) span $\{f_\alpha\}$ is the tangent space $T\mathcal{F}$ to a **foliation**: Frobenius condition $[f_\alpha, f_\beta] \in T\mathcal{F}$. Equivalently, $d\left(e^1 \wedge ... \wedge e^q\right) = -\kappa^\perp \wedge e^1 \wedge ... \wedge e^q$ for a one-form $\kappa^\perp$

(2) The **foliation is Riemannian and the metric is bundle-like**: there is a choice $\left\{e^k\right\}$ such that $f_\alpha \lrcorner de^k = 0$ for all α, k.

(3) **The mean curvature is basic**: $\sum_\alpha f_\beta \left(\omega^k_{\alpha\alpha}\right)$ for all β, k, or $f_\alpha \lrcorner d\kappa = 0$ for all α.



$$e^k = \sum_j f_j(y)\, dy^j$$

Example 1: A generic surface in $\mathbb{R}^3$

Let $z = f(x, y)$, a surface given as the graph of a function, inside Euclidean $\mathbb{R}^3$. Note that we can make this into a foliation of $\mathbb{R}^3$, by looking at the family of surfaces $z = f(x, y) + c$, as $c \in \mathbb{R}$ varies. We could write this parametrically by $F(x, y) = (x, y, f(x, y))$. Then a vector basis of the tangent space at a point is $\{U, V\}$, where

$$\begin{aligned} U &= F_x = \partial_x + f_x\partial_z = (1, 0, f_x) \\ V &= F_y = \partial_y + f_y\partial_z = (0, 1, f_y) \end{aligned}$$

(note these are not orthonormal). And an upward normal vector to the surface would be

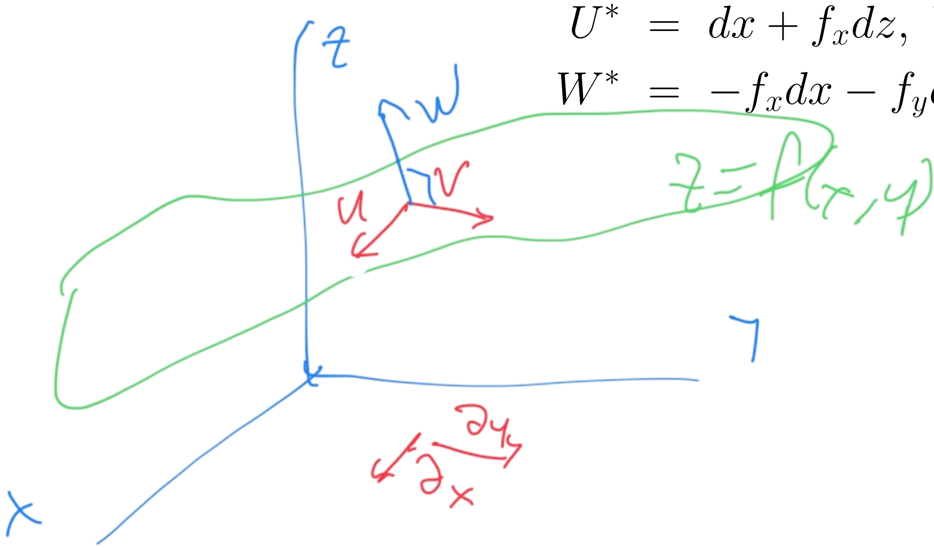
$$W = -f_x\partial_x - f_y\partial_y + \partial_z$$

$$\frac{W^*}{|W|} = e_1$$

(again, not normalized)

Corresponding covectors in $T^*\mathcal{F}$, $N^*\mathcal{F}$ would be

$$\begin{aligned} U^* &= dx + f_x dz, \ V^* = dy + f_y dz, \\ W^* &= -f_x dx - f_y dy + dz \end{aligned}$$



Then the characteristic form χ is a function η times $U^* \wedge V^*$:

$$\chi = \eta \left(dx \wedge dy + f_x dz \wedge dy + f_y dx \wedge dz\right),$$

with

$$\eta = \left(1 + f_x^2 + f_y^2\right)^{-1/2}$$

Note the normalized conormal vector is

$$\eta W^* = \eta \left(-f_x dx - f_y dy + dz\right)$$

Basic functions are functions of $(z - f(x, y))$.

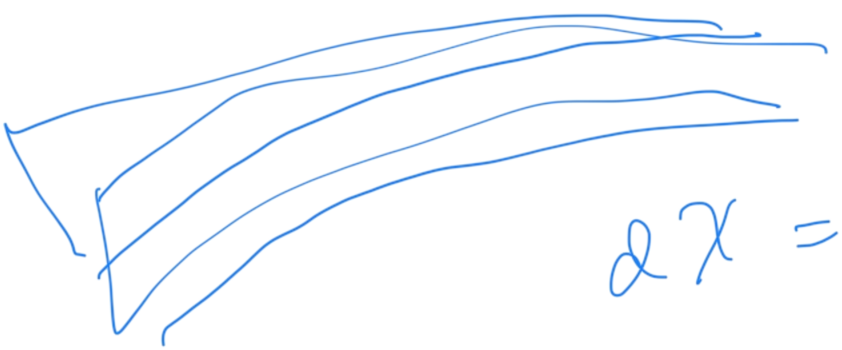
We check that

$$\boxed{d\chi = \left(-\left(\eta f_x\right)_x - \left(\eta\right) f_y)_y\right) dx \wedge dy \wedge dz,}$$

which gives from $d\chi = -\kappa \wedge \chi + \varphi$ 0

$$\begin{aligned} \kappa &= \left(\left(\eta f_x\right)_x + \left(\eta f_y\right)_y\right) W^* \\ \varphi &= 0 \end{aligned}$$

η



$d\chi = d\eta \wedge (\text{stuff})$
$+ \eta\, d(\text{stuff})$

Thus the surface is minimal iff

$$(\eta f_x)_x + (\eta f_y)_y = 0$$

(compare to minimal surface equation). Note that this foliation is not Riemannian for the standard metric except under strong conditions on f.

$$\eta = \left(1 + f_x^2 + f_y^2\right)^{-1/2}$$

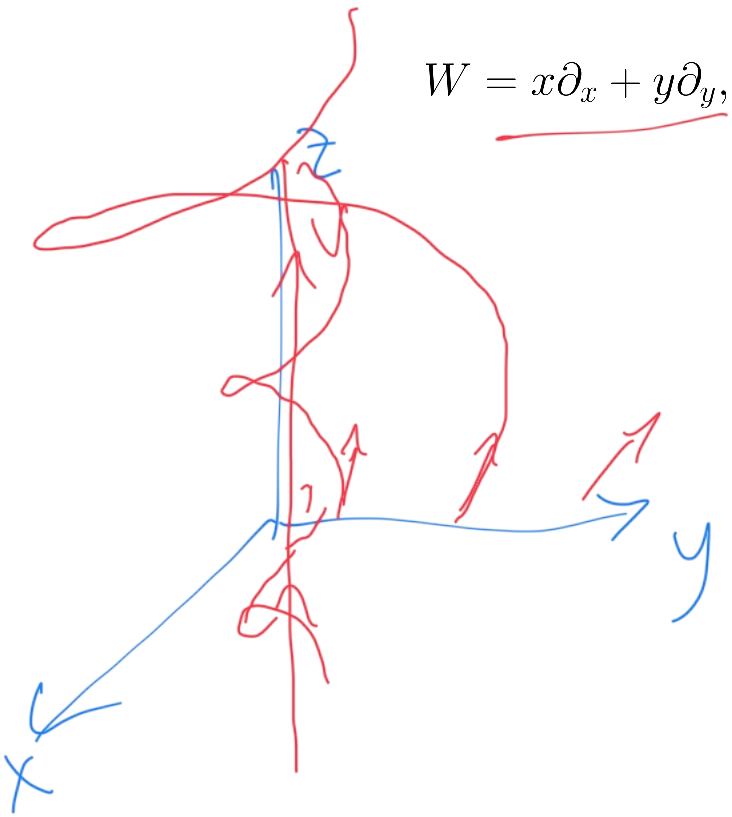
Example 2: Helixes in $\mathbb{R}^3$

Inside Euclidean $\mathbb{R}^3$, consider the vector field $V = \overbrace{-y\partial_x + x\partial_y}^{\partial_\theta} + \partial_z$. This vector field is tangent to the helixes $\alpha(t) = (r\cos(t), r\sin(t), t + c)$ for constants $r \geq 0$, $c \in \mathbb{R}$. These are the orbits of the one-parameter family of isometries

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}.$$

An orthogonal basis for the normal bundle $N\mathcal{F}$ is

$$W = x\partial_x + y\partial_y, \qquad U = y\partial_x - x\partial_y + \left(x^2 + y^2\right)\partial z.$$



Then the characteristic form of this foliation is

$$\chi = \frac{1}{|V|}V^* = \alpha\left(-ydx + xdy + dz\right),$$

with$\alpha = \left(x^2 + y^2 + 1\right)^{-1/2}, |V| = \left(x^2 + y^2 + 1\right)^{1/2}.$

Covectors normal to the leaves away from the z-axis are:

$$\begin{aligned} W^* &= xdx + ydy, \text{ normalized } \frac{1}{|W|}W^* = \frac{1}{\sqrt{x^2+y^2}}(xdx+ydy) \\ U^* &= ydx - xdy + \left(x^2+y^2\right)dz, \\ \text{normalized } \frac{1}{|U|}U^* &= \frac{1}{\sqrt{(x^2+y^2)\left(x^2+y^2+1\right)}}\left(ydx - xdy + \left(x^2+y^2\right)dz\right), \end{aligned}$$

so $|U| = |V|\ |W|$

Then

$$\begin{aligned}
d\chi &= \frac{1}{\alpha} d\alpha \wedge \chi + \alpha \left(2dx \wedge dy\right) \\
d\alpha &= -\left(xdx + ydy\right)\left(x^2 + y^2 + 1\right)^{-3/2} = -\alpha^3 W^* \\
W^* \wedge U^* &= -\left(x^2 + y^2\right) dx \wedge dy + \left(x^2 + y^2\right)\left(xdx + ydy\right) \wedge dz \\
W^* \wedge V^* &= \left(xdx + ydy\right) \wedge \left(-ydx + xdy + dz\right) \\
&= \left(x^2 + y^2\right) dx \wedge dy + \left(xdx + ydy\right) \wedge dz \\
W^* \wedge U^* - \left(x^2 + y^2\right) W^* \wedge V^* &= -\left(x^2 + y^2\right)\left(x^2 + y^2 + 1\right) dx \wedge dy \\
W^* \wedge U^* - |W|^2 W^* \wedge V^* &= -|W|^2 |V|^2 dx \wedge dy
\end{aligned}$$

Then

$$
\begin{aligned}
d\chi &= -\frac{1}{|V|^2}W^* \wedge \chi - \frac{2}{|V|}\left(\frac{1}{|W|^2\,|V|^2}W^* \wedge U^* - \frac{1}{|V|^2}W^* \wedge V^*\right) \\
&= \frac{1}{|V|^2}W^* \wedge \chi - 2\frac{1}{|W|^2\,|V|^3}W^* \wedge U^*,
\end{aligned}
$$

so that

$$
\begin{aligned}
\kappa &= -\frac{xdx + ydy}{(x^2 + y^2 + 1)} \\
\varphi &= -2\frac{1}{|W|^2\,|V|^3}W^* \wedge U^* \\
&= \frac{2}{(1 + x^2 + y^2)^{3/2}}\left(dx \wedge dy - (xdx + ydy) \wedge dz\right)
\end{aligned}
$$

$-\dfrac{r\,dr}{(r^2+1)}$

This metric is bundle-like and has basic mean curvature.

$$\kappa = -\frac{r dr}{(r^2+1)} = -\frac{1}{2} d\left(\log\left(r^2+1\right)\right)$$

exact differential one-form.

$\sqrt{x^2+y^2} = r =$ constant on each helix

2. Metric modification

Given two metrics g_1 and g_2 on a Riemannian manifold M, $tg_1 + (1-t)\,g_2$ is another Riemannian metric for $0 \le t \le 1$. (Check that we get a positive definite inner product on each T_xM, for all t.)

But what if it has additional structure. Are the structures preserved? Not necessarily. For instance, suppose g_1 and g_2 are two bundle-like metrics for the foliation $(M, \mathcal{F})$. It is possible that $tg_1 + (1-t)\,g_2$ is not a bundle-like metric.

(M, g_1), (M, g_2)

$g_t = t g_2 + (1-t) g_1$

Interestingly enough, however, it turns out that if g^1 and g^2 are the corresponding dual metrics on T^*M, then it is true that $tg^1 + (1-t)\, g^2$ is the dual metric of a Riemannian foliation, for $0 \leq t \leq 1$.

$(M, g_1, F) \longrightarrow (M, g_2, F)$

foliation

Riem foliation

Riem | foliation

Does not work $\longrightarrow$ $tg_2 + (1-t)g_1$ $\longleftarrow$ Not Riemannian foliation $\forall t$.

What does work?

$g_1 \longrightarrow g_1^* \longleftarrow$ induced metric on T^*M

$\langle \partial_x, \partial_y \rangle$

$\langle dx, dy \rangle$

$tg_2^* + (1-t)\, g_1^*$

$\searrow g_t$ – Riem fol $\forall t$.

$\langle \partial_{x_i}, \partial_{x_j} \rangle = g_{ij}$

$\langle dx_i, dx_j \rangle = g^{ij} = (g^{-1})_{ij}$