FOLIATIONS, METRICS, AND MEAN CURVATURE

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Abstract: Given a Riemannian manifold, we suppose that we are given a foliation on the manifold, i.e. a layering of immersed submanifolds. We will discuss the mean curvature vector fields and dual one-forms associated to this structure and explain their meaning. We show how to modify one metric into any other, and we quantify how the modification affects the mean curvature. Part of this talk contains joint work with Igor Prokhorenkov and Marco Radeschi.

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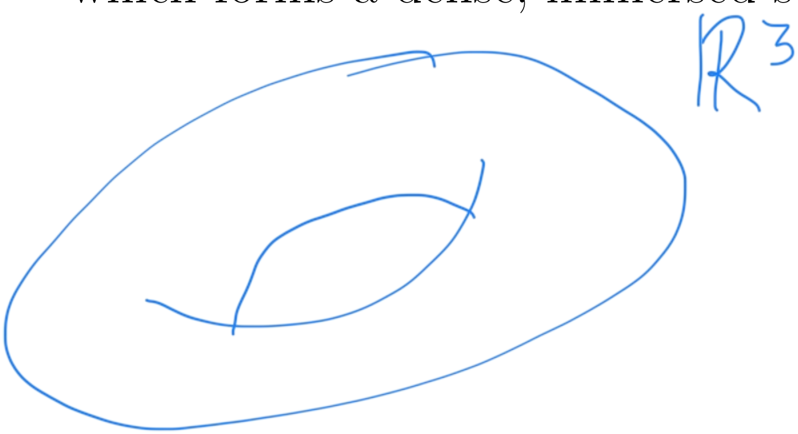
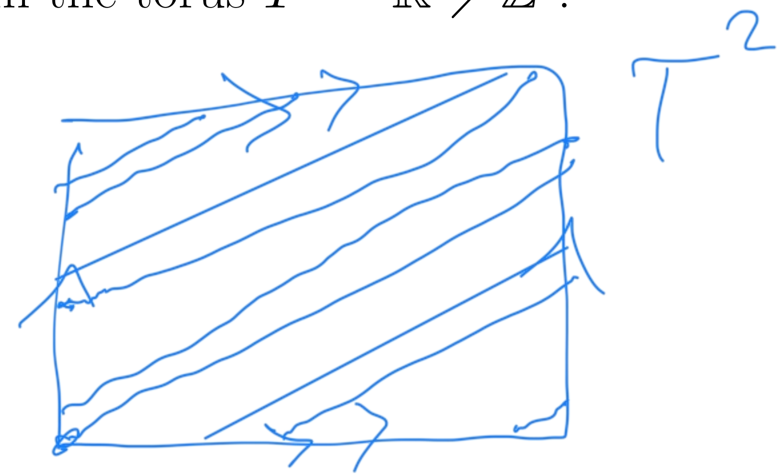
1. What is mean curvature?

Let (M, g) be a Riemannian manifold of dimension n. Many of the computations we will use are local, and in that case there is no reason to assume M is closed (compact and without boundary). When we start talking about global quantities like cohomology, we will probably restrict to the closed manifold case.

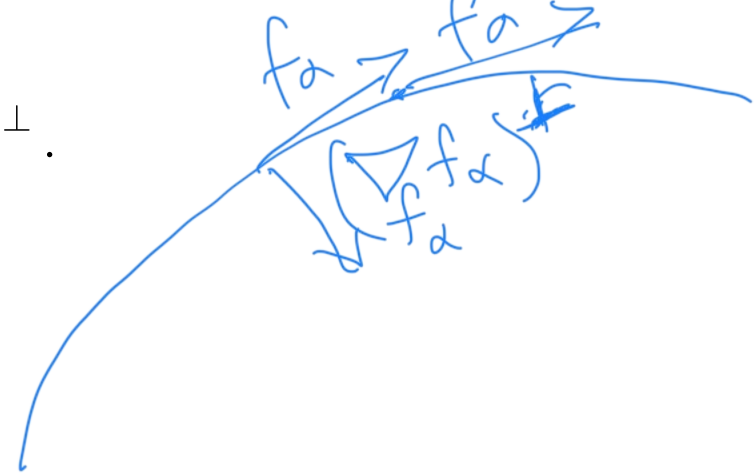
Let N be a submanifold of M of dimension p. We do not necessarily assume it is embedded, so for example, it could be the set $\left\{(x, y) = \mathbb{R}^2 \big/ \mathbb{Z}^2 : x = \sqrt{2}y\right\}$, which forms a dense, immersed submanifold in the torus $T^2 = \mathbb{R}^2 \big/ \mathbb{Z}^2$.





Let $\{f_\alpha\}_{1\leq\alpha\leq p}$ be a local orthonormal frame of $N \subseteq M$. Then we can take the covariant derivative $\nabla_{f_\alpha} f_\alpha$ in direction f_α and compute its component $(\nabla_{f_\alpha} f_\alpha)^\perp$ orthogonal to the submanifold N. The mean curvature vector field H of N is defined to be the vector field on M defined locally along N that is given by the formula

$$H = \sum_{\alpha=1}^{p} (\nabla_{f_\alpha} f_\alpha)^\perp .$$



We could check that the definition does not depend on the choice of orthonormal frame. If the local vector fields is extended in any way to be defined in an M-neighborhood of the point of N, we would get the same result when restricted to N. The submanifold N is called **minimal** if $H = 0$ at all points.

The mean curvature is the force vector of surface tension, up to a constant. Note that by rescaling the metric by a constant c^2 (shrinking N and M), the mean curvature vector field gets multiplied by $\frac{1}{c^2}$.

The mean curvature form κ is the one-form along N defined as $\kappa(v_x) = \langle v_x, H_x \rangle$ for $x \in N$, $v_x \in T_x M$; i.e. we say $\kappa = H^{\flat}$. Again, this can be extended in an M-neighborhood. This form contains the same information as H but does not change when the metric is rescaled. It is still true that $\kappa = 0$ if and only if the submanifold N is minimal.

In $\mathbb{R}^3$

$V = xy\,\partial_x + 3\,\partial_y$

$= (xy, 3, 0)$ vector in $\mathbb{R}^3$

dual $V^{\flat} = xy\,dx + 3\,dy$

The form κ can be computed using **Rummler's formula:**

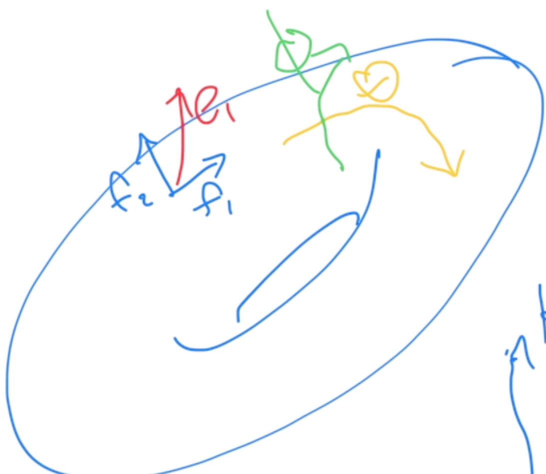
$$d\chi = -\kappa \wedge \chi + \varphi,$$

where χ is the volume form along the submanifold (extended in a neighborhood of N, also called **characteristic form**), and φ is a $(p+1)$-form on a neighborhood of N such that $v_1 \lrcorner v_2 \lrcorner ... v_p \lrcorner \varphi = 0$ on N whenever all of the vectors $v_1, ..., v_p$ are tangent to N. For a k-form α, if v is a vector at a point x, then $v \lrcorner \alpha$ is the element of $\wedge^{k-1} T_x^* M$ defined by $v \lrcorner \alpha = \alpha(v, \cdot, ..., \cdot)$. So the condition on φ says roughly that φ has at most $p-1$ "directions" along N. The pieces of this formula may be identified as the $(1,p)$ and $(2,p-1)$ components of $d\chi$, where the first index indicates the number of components normal to the submanifold direction and the second index indicates the directions in the submanifold direction.

Rummler's formula is most often used in the context of a foliation (layering of submanifolds like N so that locally the manifold if diffeomorphic to a product of manifolds). In this case, we may locally choose an adapted orthonormal frame $f_1, ..., f_p, e_1, ..., e_q$ for the tangent bundle, where $\{f_1, ..., f_p\}$ spans the tangent space $T\mathcal{F}$ to the "leaves" (local submanifolds) and $\{e_1, ..., e_q\}$ spans the normal bundle $N\mathcal{F}$ to the leaves. Then the duals of these vector fields

$f^1, \ldots, f^p, e^1, \ldots, e^q$ form a local adapted orthonormal basis for the cotangent bundle $T^*M = T\mathcal{F} \oplus N\mathcal{F}$. This could also certainly be done for any distribution of rank p, i.e. a subbundle of the tangent bundle that it is not necessarily integrable. With this adapted coframe, the characteristic form χ satisfies

$$\chi = f^1 \wedge f^2 \wedge \ldots \wedge f^p.$$



$\chi = f_1 \wedge f_2 = \square\, d\theta \wedge d\phi$

2-d distribution

$K = \square\, e_1$

Aside: If we use the Levi-Civita connection to calculate covariant derivatives of elements of an orthonormal frame $\{v_j\}$ and coframe $\{v^j\}$, we obtain the formula

$$\nabla_{v_j} v_k = \sum_s \omega_{jk}^s v_s,$$

for some functions ω_{jk}^s dependent on the metric, with symmetry $\omega_{jk}^s = -\omega_{js}^k$. Using properties of tensor derivations, we obtain corresponding derivatives for coframe elements:

$$\nabla_{v_j} v^k = -\sum_s \omega_{js}^k v^s = \sum_s \omega_{jk}^s v^s.$$

We note also that this formula can be used to calculate the differentials of these one-forms:

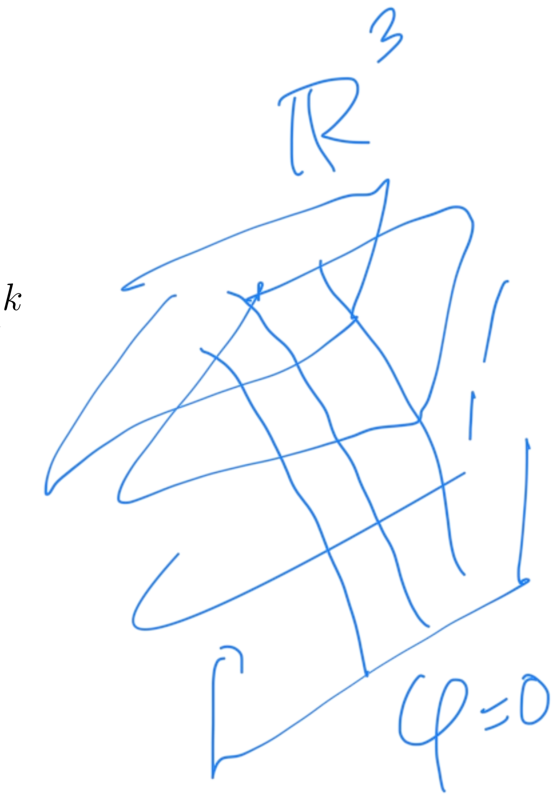
$$d\left(v^j\right) = \sum_i v^i \wedge \nabla_{v_i} v^j = -\sum_{i,k} \omega_{ik}^j v^i \wedge v^k.$$

Now, suppose a distribution of rank p has a local adapted orthonormal frame $f_1, ..., f_p, e_1, ..., e_q$ with corresponding adapted coframe $f^1, ..., f^p, e^1, ..e^q$. We will use Greek indices to indicate the f. indices, and Roman indices to indicate the e. indices. Then we have

$$\begin{aligned}
\chi &= f^1 \wedge f^2 \wedge \ldots \wedge f^p \\
d\chi &= \sum_{\alpha} (-1)^{\alpha+1} f^1 \wedge \ldots \wedge df^\alpha \wedge \ldots \wedge f^p \\
&= \sum_{\alpha} (-1)^{\alpha+1} f^1 \wedge \ldots \wedge \left(\sum_{k,\beta} \left(-\omega^\alpha_{k\beta} e^k \wedge f^{\overset{=\alpha}{\beta}} - \omega^\alpha_{\beta k} f^{\overset{=\alpha}{\beta}} \wedge e^k\right) - \sum_{r,s} \omega^\alpha_{rs} e^r \wedge e^s \right) \\
&\qquad \wedge \ldots \wedge f^p \\
&= \underbrace{\sum_{\alpha,k} \left(\omega^\alpha_{\alpha k} - \omega^\alpha_{k\alpha}\right) e^k}_{-k} \wedge \underbrace{\chi}_{p} + \left(\sum_{\alpha,r,s} (-1)^\alpha \omega^\alpha_{rs} \underbrace{e^r \wedge e^s}_{2} \wedge \underbrace{f^1 \wedge \ldots \wedge \widehat{f^\alpha} \wedge \ldots \wedge f^p}_{p-1} \right) \\
&= -\kappa \wedge \chi + \varphi,
\end{aligned}$$

where the two pieces of the formula have the desired properties, with κ of type $(1, 0)$ and φ of type $(2, p-1)$.

Furthermore



$$\begin{aligned}
\kappa &= \sum_{\alpha,k} \left(\omega^{\alpha}_{k\alpha} - \omega^{\alpha}_{\alpha k}\right) e^k = \sum_{\alpha,k} \left(0 - \omega^{\alpha}_{\alpha k}\right) e^k \\
&= \sum_{\alpha,k} \omega^{k}_{\alpha\alpha} e^k = \left(\sum_{\alpha,k} \omega^{k}_{\alpha\alpha} e_k \right)^{\flat} \\
&= \left(\sum_{\alpha} \left(\nabla_{f_\alpha} f_\alpha\right)^{\perp} \right)^{\flat} = H^{\flat}.
\end{aligned}$$

We note also that the form φ is zero if and only if the normal bundle to the distribution is involutive (i.e. forms the tangent space to a foliation).

The following are other equivalent formulas for mean curvature:

$$\begin{aligned}
\kappa &= (-1)^{p+1}\,\chi\lrcorner d\chi, \\
\kappa &= \sum_{\alpha,k}\left(f_\alpha\lrcorner df^\alpha, e^k\right) e^k = \sum_{\alpha,k}\left(e_k\lrcorner f_\alpha\lrcorner df^\alpha\right) e^k, \\
\kappa &= \sum_{\alpha} f_\alpha\lrcorner d_{1,0} f^\alpha, \\
\kappa &= -\sum_{\alpha,k}(f_\alpha,[f_\alpha,e_k])e^k.
\end{aligned}$$

Local conditions on the distribution:

(1) span $\{f_\alpha\}$ is the tangent space $T\mathcal{F}$ to a **foliation**: Frobenius condition $[f_\alpha, f_\beta]\in T\mathcal{F}$. Equivalently, $d\left(e^1\wedge\ldots\wedge e^q\right) = -\kappa^\perp\wedge e^1\wedge\ldots\wedge e^q$ for a one-form $\kappa^\perp$

(2) The **foliation is Riemannian and the metric is bundle-like**: there is a choice $\left\{e^k\right\}$ such that $f_\alpha\lrcorner de^k = 0$ for all α, k.

(3) **The mean curvature is basic**: $\sum_\alpha f_\beta\left(\omega^k_{\alpha\alpha}\right)$ for all β, k, or $f_\alpha\lrcorner d\kappa = 0$ for all α.

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