INTRODUCTION TO SINGULAR RIEMANNIAN FOLIATIONS

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1. Foliations

A (smooth) **foliation** $\mathcal{F}$ of a smooth manifold M is a partition of M complete, connected, immersed submanifolds (leaves) of the same dimension such that for all $x \in M$, there exists a distinguished neighborhood N of x such that $N \cong \mathbb{R}^p \times \mathbb{R}^q$, where each $\mathbb{R}^p \times \{u\}$ corresponds to a subset (called a **plaque**) of a leaf. The set $\mathcal{F}$ is the collection of leaves, and $L = T\mathcal{F} \subseteq TM$ denotes the tangent bundle to $\mathcal{F}$. Further, there is a compatibility condition on the overlaps: the transition functions $\varphi_{ij} : \mathbb{R}^p \times \mathbb{R}^q$ take the form

$$\varphi_{ij}(x, y) = \left(\varphi_{ij}^1(x, y), \varphi_{ij}^2(y)\right)$$

where φ_{ij}^2 is a diffeomorphism of $\mathbb{R}^q$ and for fixed y, $\varphi_{ij}^1(\cdot, y)$ is a diffeomorphism of $\mathbb{R}^p$.

Example 1.1. *Let $M = Y \times F$ be a product manifold. Then the sets $\{y\} \times F$ form a foliation of M.*

Example 1.2. *Let $M = T^2 = \mathbb{R}^2 \diagup \mathbb{Z}^2$, the torus, and let $T\mathcal{F}$ be the vectors of slope m in the tangent spaces, so that $\mathcal{F}$ is the set of lines of slope m. Then $\mathcal{F}$ is a foliation. If m is irrational, then each leaf is dense in M. Note also that the subspace topology of each leaf is a topology on $\mathbb{R}$ that is not the standard topology. If m is rational, then each leaf is a circle.*

Example 1.3. *Consider the suspension of an irrational rotation ϕ of the sphere S^2. This means $S^2 \times \mathbb{R} \diagup \sim$, where $(p, t) \sim (\phi(p), t + 1)$. The leaves are of the form $\{p\} \times \mathbb{R} \diagup \sim$, so that each leaf is dense in the (latitude) $\times \mathbb{R} \diagup \sim$, which is a torus away form the poles and is a circle at the poles.*

Example 1.4. *Let G be a compact Lie group that acts on X by isometries. If all the orbits of G have the same dimension, then the orbits form a foliation of X.*

Example 1.5. *On $T^2 = \mathbb{R}^2 \diagup \mathbb{Z}^2$, let the foliation be the union of the circle $\{0\} \times \mathbb{R} \diagup \mathbb{Z}^2$ and leaves in the open set $(0, 1) \times \mathbb{R} \diagup \mathbb{Z}^2$ of the form $\left\{\left(x, \frac{1}{x} + \frac{1}{1-x} + c\right) : x \in (0, 1)\right\} \diagup \mathbb{Z}^2$ for fixed $c \in \mathbb{R}$. Note that the closure of each noncompact leaf is itself union the circular leaf. We call this a* ***Reeb-type foliation****.*

Note that a foliation is more than the choice of a subbundle (distribution) L of the tangent bundle TM. The famous **Frobenius theorem** states that a distribution L of TM is the tangent bundle to a foliation $\mathcal{F}$ on M if and only if $[\Gamma L, \Gamma L] \subseteq \Gamma L$; that is, the bracket of any two vector fields V, W such that $V_x, W_x \in L_x$ for every x also satisfies $[V, W]_x \in L_x$. A distribution L satisfying the condition $[\Gamma L, \Gamma L] \subseteq \Gamma L$ is said to be **involutive**.

2. Riemannian foliations

A Riemannian foliation is a smooth foliation endowed with a metric g_Q on the quotient bundle $Q = TM \diagup T\mathcal{F}$ such that this metric can be extended to a **bundle-like** metric g on

M with g_Q consistent with g restricted to the normal bundle $N\mathcal{F}$ of the foliation. The word **bundle-like** means that the Lie derivative $\mathcal{L}_X g^{\perp}$ of the transverse metric on $N\mathcal{F}$ is zero for any leafwise vector fields X. Other equivalent conditions to this are that the normal bundle $N\mathcal{F}$ is totally geodesic, or that a geodesic that is normal to a leaf at one point is normal to all leaves that it meets. Another intuitive way of understanding such a metric is that these requirements are equivalent to requiring that the leaves are locally equidistant.

Not every foliation can be given a metric so that it is a Riemannian foliation: there are topological obstructions. For instance, in Riemannian foliations, the leaf closures partition the manifold and are in particular disjoint. This does not happen in Example 1.5, so the Reeb-type foliation is not Riemannian. However, the other four examples in the previous section are Riemannian foliations in the obvious metrics.

Riemannian foliations are foliations with metrics for which it is possible to do some global analysis and geometric analysis. They were introduced by Reinhart in [22] (see also [23]).

One example of a Riemannian foliation is obtained by looking at the orbits of a compact Lie group action on a manifold such that all the orbits have the same dimension. Note that in this case the entire metric is preserved in the leaf direction, i.e. $\mathcal{L}_X g = 0$ for all leafwise X. For flows (1-dimensional foliations), the flow comes from an isometric flow in some metric if and only if the foliation is taut (i.e. there exists a metric for which the leaves are minimal submanifolds). However, the Álvarez class $[\kappa_b]$ in basic cohomology H_b^1 is an obstruction to this; here κ_b is the basic component of the mean curvature 1-form, which is always closed.

Here is an example of a Riemannian flow which is not isometric for any metric.

Example 2.1. *We consider the Carrière example from* [10] *in the* 3*-dimensional case. Let* A *be a matrix in* $\mathrm{SL}_2(\mathbb{Z})$ *of trace strictly greater than* 2*. We denote respectively by* V_1 *and* V_2 *the eigenvectors associated with the eigenvalues* λ *and* $\frac{1}{\lambda}$ *of* A *with* $\lambda > 1$ *irrational. Let the hyperbolic torus* $\mathbb{T}_A^3$ *be the quotient of* $\mathbb{T}^2 \times \mathbb{R}$ *by the equivalence relation which identifies* (m,t) *to* $(A(m), t+1)$*. The flow generated by the vector field* V_2 *is Riemannian. We choose the bundle-like metric (letting* (x,s,t) *denote the local coordinates in the* V_2 *direction,* V_1 *direction, and* $\mathbb{R}$ *direction, respectively) as*

$$g = \lambda^{-2t}dx^2 + \lambda^{2t}ds^2 + dt^2.$$

Notice that the mean curvature of the flow is $\kappa = \kappa_b = \log(\lambda)\, dt$*, since* $\chi_{\mathcal{F}} = \lambda^{-t}dx$ *is the characteristic form and* $d\chi_{\mathcal{F}} = -\log(\lambda)\,\lambda^{-t}dt \wedge dx = -\kappa \wedge \chi_{\mathcal{F}}$*. The cohomology group* $H_b^1(M, \mathcal{F})$ *is generated by* $[\kappa_b]$*.*

3. Smooth Singular Foliations

A **smooth singular** distribution L on a smooth manifold M is a collection $\{L_x : x \in M\}$ where for each x, L_x is a subspace of T_xM, and such that for every $x \in M$, there exists a neighborhood U containing x such that $L|_U$ is locally spanned by smooth vector fields. That is, there exists a collection $\{V_\alpha\}_{\alpha\in A}$ of smooth vector fields on U such that

$$L_x = \operatorname{span}\{V_\alpha(x) : \alpha \in A\}$$

for each $x \in U$.

A **smooth singular foliation** is a partition of M into immersed submanifolds such that the tangent bundle $T\mathcal{F}$ is a smooth singular distribution. The smoothness of singular distributions was first developed independently by Stefan and Sussman in [26], [27]. It turns out that there is a natural generalization of the Frobenius theorem to the setting of singular distributions/foliations — essentially the same statement. Incidentally, with Drager, Lee, and Park, I proved that a smooth singular distribution on a smooth manifold is always globally finitely generated, meaning that there always exists a finite set of vector fields on the manifold such that the distribution is exactly the span of those vector fields (see [12]).

Example 3.1. *Let V be any smooth vector field on a smooth manifold M. Then the orbits of V form a smooth singular foliation of M.*

Example 3.2. *Let a compact Lie group G act on a smooth manifold M. Then the orbits of G in M form a smooth singular foliation of M.*

Given a singular foliation, a criterion that can be used to determine if it is smooth is the following.

Proposition 3.3. *Let $(M, \mathcal{F})$ be a singular foliation, so that M is partitioned into complete, connected (immersed) submanifolds in $\mathcal{F}$. Then M is smooth if and only if every vector in $T\mathcal{F}$ can be locally extended to a vector field that is smooth and is everywhere tangent to $T\mathcal{F}$.*

4. Singular Riemannian foliations

An **SRF**, a **singular Riemannian foliation** $(M, g, \mathcal{F})$ is a smooth singular foliation $\mathcal{F}$ on a Riemannian manifold (M, g) that satisfies the metric condition that geodesics orthogonal to the leaves at one point are orthogonal to the leaves at any point. Sometimes, if the metric is understood, we will simply use $(M, \mathcal{F})$ to refer to the singular Riemannian foliation.

Note that if $\mathcal{F}$ is simply a partition of the Riemannian manifold (M, g) into complete connected immersed submanifolds such that geodesics orthogonal to the leaves at one point are orthogonal to the leaves at any point, $(M, g, \mathcal{F})$ is called a **transnormal system**. So a transnormal system is an SRF if in addition every leafwise vector can be extended locally to a leafwise vector field.

Conjecture 4.1. *(Folk Conjecture, according to Radeschi, still open) Every transnormal system is an SRF.*

If all the leaves of $\mathcal{F}$ have the same dimension, $\mathcal{F}$ is called regular. In this case, the condition on g given above is equivalent to g being a bundle-like metric on a Riemannian foliation.

Let the stratum $\Sigma_r \subseteq M$ denote the union of leaves of dimension r. Then the restriction of $\mathcal{F}$ and g to each Σ_r is a Riemannian foliation with bundle-like metric. The stratum corresponding to leaves of the smallest dimension is a compact submanifold, called the **minimal stratum**. The stratum corresponding to leaves of maximal dimension is open and dense in M and is called the **regular stratum**. The closures of the leaves of a singular Riemannian foliation are submanifolds, and the restriction of $\mathcal{F}$ to one of these leaf closures is a [transversally locally homogeneous] regular Riemannian foliation.

A **singular Riemannian flow** is a singular Riemannian foliation such that the maximal dimension of each leaf is one.

We say that a smooth vector field X on a smooth manifold M is a **transverse Killing vector field** if there exists a Riemannian metric on M such that the singular flow generated by X is a singular Riemannian flow. If the zero set Σ of X is nondegenerate, meaning the normal Hessian of X is invertible at Σ, we say that X is a **nondegenerate transverse Killing vector field**. One can always construct a nondegenerate transverse Killing vector field corresponding to any oriented singular Riemannian flow. We remark that in other sources the term "transverse Killing" implies a choice of metric on the normal bundle to the foliation, but we do not specify this metric in our definition.

Example 4.2. *Let a compact Lie group G act on a smooth manifold M. Then the orbits of G in M form a smooth singular foliation of M. Now let $\langle\ ,\ \rangle$ be any metric on M, and let $\overline{\langle\ ,\ \rangle}$ be the metric averaged over G; then with this new metric, the orbits of G in M form a singular Riemannian foliation. In symbols, if $v, w \in T_xM$, let g_*v, g_*w denote the push-forwards of v, w to $T_{gx}M$, so the definition of $\overline{\langle\ ,\ \rangle}$ is*

$$\overline{\langle v, w\rangle}_x = \int_G \langle g_*v, g_*w\rangle_{gx}\ dg,$$

where dg is the bi-invariant volume form on G such that $\int_G dg = 1$.

Example 4.3. *Let $(M, \mathcal{F})$ be a regular Riemannian foliation of a compact manifold. Then the leaf closures partition the manifold and form an SRF $\left(M, \overline{\mathcal{F}}\right)$. This theorem is due to Molino (*[18]*).*

5. Polar foliations

One particular example of SRFs is what is called a **polar foliation**.

A **polar action** is isometric Lie group actions such that through every point there is a submanifold, called a **section**, that meets every orbit orthogonally and the principal orbits transversally. This is equivalent to the integrability of the normal bundle on the principal stratum by Theorem 5.1 below.

A typical example of a polar action is the action of a compact Lie group on itself by conjugation; the maximal tori are the sections. More generally the isotropy actions on symmetric spaces are polar. A very similar structure can be found in submanifold theory. The decomposition by parallel submanifolds of an isoparametric submanifold in the sphere or in Euclidean space demonstrate similar properties. Now both these classes are special cases of **polar foliations**, or **SRFs admitting sections** (the definition of a section is essentially the same). In fact, some of the properties that are shared by polar actions and isoparametric foliations can be derived for polar foliations. The slice theorem for polar actions can be strengthened (see below). This states that the isotropy action on a slice of an orbit is again a polar action. Likewise the slice theorem for polar SRFs can be strengthened to say that the restriction of a polar foliation to a slice is again a polar foliation.

A singular Riemannian foliation $(M, g, \mathcal{F})$ is called a **polar foliation** (or a **singular Riemannian foliation with sections**) if, for each regular point p (point of the principal stratum), there is an immersed submanifold Σ_p through p, called a **section**, whose dimension is equal to the codimension of the foliation and that meets all the leaves perpendicularly. It follows that Σ_p is totally geodesic. This is equivalent to the normal bundle $N\mathcal{F}$ restricted to the principal stratum of $(M, \mathcal{F})$ is integrable.

An **integrable singular Riemannian foliation (ISRF)** is an SRF such that normal bundle is integrable on the principal stratum.

It is natural to ask if an ISRF is a polar foliation. The next result gives a positive answer to this question (and also answers the simpler question for polar actions).

Theorem 5.1. *(in* [2]*) Let $(M, g, \mathcal{F})$ be an ISRF on a complete Riemannian manifold. Then $\mathcal{F}$ is a polar foliation, and regular points are open and dense in each section.*

Polar group actions provide one particular example of polar foliations. Another important class of examples is the partition of a Euclidean space into the parallel submanifolds of an isoparametric submanifold N. Recall that a submanifold N of a Euclidean space is called **isoparametric** if its normal bundle is flat and the principal curvatures along any parallel normal vector field are constant. [Note that there are examples of inhomogeneous isoparametric submanifolds.] In the slice theorem below, if $(M, g, \mathcal{F})$ is polar then the infinitesimal foliation $\widehat{\mathcal{F}}(q)$ is polar and hence isoparametric.

6. Molino Conjecture

Theorem 6.1. *(Molino, 1980s,* [18]*) Given a SRF $(M, g, \mathcal{F})$, let $\overline{\mathcal{F}}$ be the collection of leaf closures of $\mathcal{F}$. Then $\overline{\mathcal{F}}$ partitions M and forms a transnormal system.*

Theorem 6.2. *(Molino, 1980s,* [18]*) Given a (nonsingular) Riemannian foliation $(M, g, \mathcal{F})$, let $\overline{\mathcal{F}}$ be the collection of leaf closures of $\mathcal{F}$. Then $\overline{\mathcal{F}}$ partitions M and forms a SRF.*

Conjecture 6.3. *(Molino, 1980s: **The Molino Conjecture**) Given a SRF $(M, g, \mathcal{F})$, let $\overline{\mathcal{F}}$ be the collection of leaf closures of $\mathcal{F}$. Then $\overline{\mathcal{F}}$ partitions M and forms a SRF.*

So, as can be seen above, the gap remaining for proving the Molino conjecture is to show that $\overline{\mathcal{F}}$ is actually smooth, i.e. to show that every vector tangent to a leaf closure can be extended to a smooth vector field that is tangent to every leaf closure that it meets.

In Marcos Alexandrino and Marco Radeschi's paper [4], the authors proved the Molino conjecture for **orbit-like foliations**. A different strategy for proof of the conjecture in this case was previously suggested in [19]. There also have been proofs of the conjecture in other special cases (see [2] for the proof in the case of polar foliations).

Definition 6.4. *Given a singular Riemannian foliation $(M, g, \mathcal{F})$, the **slice foliation** at $p \in M$, denoted $\left(S_p, \mathcal{F}|_{S_p}\right)$ is defined as follows. The leaf of $\mathcal{F}$ through p is denoted L_p, and $\nu_p L_p \subseteq TM$ is the normal space to L_p at p. Given $\varepsilon > 0$, let $\nu_p^\varepsilon L_p$ denote the set of $x \in \nu_p L_p$ of norm $< \varepsilon$. If ε is small enough, the normal exponential map* $\exp : \nu_p^\varepsilon L_p \to M$ *is a diffeomorphism onto its image S_p, called the **slice** of L_p at p. The slice foliation $\mathcal{F}|_{S_p}$ is the partition of S_p into the connected components of the intersection $L \cap S_p$, with $L \in \mathcal{F}$.*

Definition 6.5. *Given a singular Riemannian foliation $(M, \mathcal{F})$, the **infinitesimal foliation** at $p \in M$, denoted $\left(\nu_p L_p, \widehat{\mathcal{F}}_p\right)$ is defined as follows. The foliation $\widehat{\mathcal{F}}_p$ is defined as the partition of $\nu_p L_p$, where the leaf at $v \in \nu_p L_p$ is*

$$L_v = \left\{ w \in \nu_p L_p : \exp_p(tw) \in L_{\exp_p(tv)} \ \forall \text{ suff. small } t > 0 \right\},$$

where $L_{\exp_p(tv)}$ denotes the leaf of the slice foliation $\left(S_p, \mathcal{F}|_{S_p}\right)$ through $\exp_p(tv)$.

Remark 6.6. *Given a singular Riemannian foliation $(M, \mathcal{F})$ and a submanifold N of M that is a union of leaves of the same dimension, the infinitesimal foliation splits as a product $\left(\nu_p(L_p, N) \times \nu_p N, \{pts\} \times \mathcal{F}_p|_{\nu_p N}\right)$, where $\nu_p(L_p, N) = \nu_p L_p \cap T_p N$. In this case, the foliation $\left(\nu_p N, \mathcal{F}_p|_{\nu_p N}\right)$ is the "essential part" of the infinitesimal foliation; by abuse of notation, sometimes this is also called the infinitesimal foliation and is denoted $\mathcal{F}_p$. Note that the origin is always a leaf of $\mathcal{F}_p$.*

A singular Riemannian foliation $(M, \mathcal{F})$ is called **homogeneous** if there exists a connected Lie group G acting by isometries on M, whose orbits are precisely the leaves of M. A singular Riemannian foliation $(M, \mathcal{F})$ is called **orbit-like** if at every point $p \in M$, the infinitessimal foliation $(\nu_p L_p, \mathcal{F}_p)$ is closed [i.e. consists of closed leaves] and is homogeneous.

Theorem 6.7. *(**Slice Theorem,** [1]) Let $(M, g, \mathcal{F})$ be a polar foliation on a complete Riemannian manifold. Let q be a singular point (I think this can be any point) of M and let S_q be a slice at q of radius ε. Then*

1. *$S_q = \bigcup_{\sigma \in A(q)} \sigma$, where $A(q)$ is the set of local sections σ containing q such that $dist\,(p, q) < \varepsilon$ for each $p \in \sigma$.*
2. *$S_x \subset S_q$ for all $x \in S_q$.*
3. *$\mathcal{F}|_{S_q}$ is a polar foliation on S_q with respect to the induced metric.*
4. *The infinitesimal foliation $\widehat{\mathcal{F}}(q)$ is polar and hence isoparametric.*

Remark 6.8. *An SRF $(M, g, \mathcal{F})$ is called **infinitesimally polar** if its infinitesimal foliation at each point is polar.*

M. Alexandrino and M. Radeschi proved the Molino conjecture in complete generality in [5].

The main theorem of the paper is the following.

Theorem 6.9. *(**Main Theorem of** [5]) Let $(M, \mathcal{F})$ be a singular Riemannian foliation. Let L be a (possibly nonclosed) leaf, and let U be an ε-neighborhood of the leaf closure of L. Then for ε small enough, there is a metric g^ℓ on U and a singular foliation $\widehat{\mathcal{F}}^\ell$ on U such that:*

1. *$\left(U, g^\ell, \widehat{\mathcal{F}}^\ell\right)$ is an orbit-like singular Riemannian foliation.*
2. *The foliation $\widehat{\mathcal{F}}^\ell$ coincides with $\mathcal{F}$ on $\overline{L}$.*
3. *The closure of $\widehat{\mathcal{F}}^\ell$ is contained in the closure of $\mathcal{F}$.*

The superscript ℓ is supposed to make us think "linearized". And then a corollary is:

Theorem 6.10. *(**Molino Conjecture Theorem,** [5]) Let $(M, \mathcal{F})$ be a singular Riemannian foliation on a complete manifold, and let $\overline{\mathcal{F}} = \left\{\overline{L} : L \in \mathcal{F}\right\}$ be the partition of M into the closures of the leaves of $\mathcal{F}$. Then $\left(M, \overline{\mathcal{F}}\right)$ is a singular Riemannian foliation.*

Proof. Molino himself proved that $\left(M, \overline{\mathcal{F}}\right)$ is a transnormal system with closed leaves, so it is enough to show that for any leaf $L \in \mathcal{F}$ with closure $\overline{L}$ and any vector $v \in \nu\left(L, \overline{L}\right) := \nu L \cap \overline{L}$, there exists a smooth extension of v to a vector field everywhere tangent to the leaves of $\overline{\mathcal{F}}$. Let U be a tubular neighborhood of $\overline{L}$, and let $\left(U, \widehat{\mathcal{F}}^\ell\right)$ be the foliation satisfying the main theorem. Since $\widehat{\mathcal{F}}^\ell$ is orbit-like, by previous work, there exists a vector field V extending v that is tangent to the closure of $\overline{\widehat{\mathcal{F}}^\ell}$. Since the closure is contained in $\overline{\mathcal{F}}$, it follows that V is also tangent to the leaves of $\overline{\mathcal{F}}$. $\square$

Thus, the above facts will follow as long as we are able to construct g^ℓ and $\widehat{\mathcal{F}}^\ell$. At each point p of a leaf closure $\overline{L}$, we construct the infinitesimal foliation $\mathcal{F}_p$. Then, restricted to $\nu_p L_p$, this is given by the orbits of a Lie group of isometries. We then take the closure of this group of isometries and the corresponding orbits to yield the potentially larger foliation

$\overline{\mathcal{F}_p}$, the "local closure" of $\mathcal{F}_p$. Using the normal exponential map, we may transplant this singular foliation to the tubular neighborhood U of $\overline{L}$. Further, the foliation $\left(\overline{L}, \mathcal{F}|_{\overline{L}}\right)$ may be extended to a distribution $\left(U, \widehat{\mathcal{F}}\right)$ of the same dimension as $\mathcal{F}$ by taking horizontal lifts to the total space of $\nu_p^{\varepsilon} L_p$. Finally, for $v \in \nu_p^{\varepsilon} L_p$ we take $T_v\widehat{\mathcal{F}} \oplus \overline{\mathcal{F}_p}$ to be a distribution transplanted to U. Then it remains to be shown that this distribution is involutive, and then we take the corresponding foliation to be $\widehat{\mathcal{F}}^{\ell}$.

The metric g^{ℓ} is defined by first splitting $T\overline{L}$ into $T\overline{L} = T\left(\mathcal{F}|_{\overline{L}}\right) \oplus \nu\left(\mathcal{F}|_{\overline{L}}\right)$ and then splitting $TU|_{\overline{L}}$ into $TU = T\left(\mathcal{F}|_{\overline{L}}\right) \oplus \nu\left(\mathcal{F}|_{\overline{L}}\right) \oplus T\left(\text{fibers of } \exp^{\perp}\right)$. The metric is then defined as a direct sum metric and is extended in a natural way to all of U. Then g^{ℓ} and $\widehat{\mathcal{F}}^{\ell}$ can be shown to have all the desired properties in the main theorem.

7. Related open questions

Problem 7.1. *Classify polar actions on symmetric spaces, or on some manifolds in general.*

This has been done for compact rank 1 symmetric spaces.

Problem 7.2. *Classify SRFs on spheres and/or Euclidean space.*

This was done by Gromoll and Grove for regular Riemannian foliations.in dimensions 1,2,3, and for SRFs in dimensions 1,2,3 by Radeschi.

Problem 7.3. *Generalize results know for isometric group actions to SRFs.*

We have one example of this in the next section.

8. Generalization of Bott's characteristic number calculation

This work [21] is joint with Igor Prokhorenkov. We establish the structure of an oriented singular Riemannian flow $(M, \mathcal{F}, g)$ in the tubular neighborhood of a component of the singular stratum $\Sigma := \Sigma_0$. This theorem resembles slice theorems (such as in [20], [3], [17]), but the new results in this paper are stronger for flows in that they apply to the entire tubular neighborhood of a singular stratum rather than to the neighborhood of a singular leaf. We show that there exists a new metric g' on M for which $(M, \mathcal{F}, g')$ is a singular Riemannian flow on M that restricts to an isometric flow on the tubular neighborhood. Note that every vector field that generates an isometric flow for some metric on M is automatically a nondegenerate transverse Killing vector field and thus generates a singular Riemannian flow. It is easy to construct transverse Killing fields that are not global Killing vector fields for any metric; equivalently, there are singular Riemannian flows that are not foliated-diffeomorphic to singular isometric flows. See Examples 8.4 and 8.5.

In the paper's main theorem, we provide the formula that computes characteristic numbers of an even-dimensional, oriented closed manifold as the sum of residues at the components of the zero set of a nondegenerate transverse Killing vector field that generates a singular Riemannian flow. We prove that the Lie derivative of the field induces an isometric flow on the normal bundle of each component of the singular stratum, and the residue at this component is defined in terms of the invariants of this action. In the case when the singular Riemannian flow is not orientable, the argument is easily handled by a modification of the theorem. These theorems specialize to the results in [7] in the case when the singular

Riemannian flow is in fact a global isometric flow for some metric. One simple consequence is the formula for the Euler characteristic,

$$\chi(M)=\sum_j \chi(\Sigma_j),$$

where Σ_j are the components of the singular stratum of a possibly nonorientable Riemannian flow $(M,\mathcal{F},g)$. This formula was previously known when Σ_j are the zero sets of a Killing vector field; see [15]. We also derive a new formula for the signature of a manifold endowed with a singular Riemannian flow whose singular stratum is a finite set of points.

The history of this problem is as follows. In the celebrated paper [8], R. Bott showed how to compute the Pontryagin and other characteristic numbers from isolated singular points of holomorphic vector fields or of infinitessimal isometries. In [9], he generalized his result in the holomorphic case to allow vector fields whose zero sets are submanifolds. In [6], M. Atiyah and I. Singer used the G-signature theorem, a special case of the index theorem, to give the formula for the characteristic numbers of a singular isometric flow in terms of integrals of characteristic forms over the singular stratum of the flow. In [7], P. Baum and J. Cheeger use purely differential-geometric and Stokes' theorem techniques to derive the same result. One consequence of all these results is that if there exists a nonvanishing Killing vector field on a closed Riemannian manifold, then all of its characteristic numbers vanish. In [11], Y. Carrière showed that any Riemannian manifold with a nonsingular Riemannian flow has Gromov minimal volume zero, so as a consequence all of the characteristic numbers of that manifold are zero, consistent with our theorem. In [16], X. Mei considered a singular Riemannian foliation and a variant of curvature coming from the curvature of the normal bundle to the foliation. The author gave a formula for the residue of a characteristic polynomial of this type of curvature at a connected component of the singular stratum. These are not the same as the residues used to compute the characteristic numbers of the manifold, which are computed in the paper with Igor.

We now introduce the notation of the main theorem, much of which is similar to that in [7, Section 1]. Given any invertible linear transformation $A\in\mathfrak{o}(2s)$, there exists an orthonormal basis $\{e_1,...,e_{2s}\}$ for $\mathbb{R}^{2s}$ such that $Ae_{2j-1}=\lambda_j e_{2j}$ and $Ae_{2j}=-\lambda_j e_{2j-1}$ and $\lambda_j\geq 0$ for each j. The numbers λ_j are called **skeigen-values**. It is well-known that if ψ is an $\mathrm{ad}(\mathrm{SO}(2s))$-invariant symmetric complex-valued polynomial on $\mathfrak{o}(2s)$, there exists a unique polynomial $\widehat{\psi}:\mathbb{R}^{s+1}\to\mathbb{C}$ such that

$$\psi(A)=\widehat{\psi}(\lambda_1,...,\lambda_s).$$

for any such transformation A. The Pfaffian $\chi(A)$ of A is a particular example; $\chi(A)=\widehat{\chi}(\lambda_1,...,\lambda_s)=\pm\lambda_1...\lambda_s$, where the positive sign is chosen exactly when $e_1,...,e_{2m}$ is a positively oriented basis of $\mathbb{R}^{2s}$.

The given nondegenerate transverse Killing field X with singular set Σ, its linearization restricts to each $N_x\Sigma$ to be a Killing field. The restriction of its Lie derivative to $N_x\Sigma$ is a nonsingular skew-symmetric automorphism $P_x\left(\mathcal{L}_X|_{\Gamma(N\Sigma)_x}\right)$, where $P_x:T_xM\to N_x\Sigma$ is the orthogonal projection. Further we multiply the endomorphism by a positive scalar c_x so that the resulting skeigen-values $\{\alpha_j\}$ satisfy $\sum\alpha_j^2=1$ and each α_j is nonzero. Let $\Lambda_X^\nu=c_xP_x\left(\mathcal{L}_X|_{\Gamma(N\Sigma)_x}\right)$. We extend Λ_X^ν by zero on $T\Sigma$ to define the endomorphism $\Lambda_X: TM|_\Sigma\to TM|_\Sigma$. The skeigen-values of Λ_X and of Λ_X^ν (i.e. the nonzero skeigen-values of Λ_X) are constant on each connected component of Σ. Let $\mu_0=0$, $\mu_1,...,\mu_\tau$ be the distinct skeigen-values of Λ_X. Furthermore, $TM|_\Sigma$ is the direct sum of skeigen-bundles $TM|_\Sigma = E_0 \oplus E_1 \oplus ... \oplus E_\tau$, where $E_0 = T\Sigma$ and

$$(E_j)_x = -\mu_j^2 \text{ eigenspace of } (\Lambda_X)_x^2$$

For each $j \geq 1$, E_{λ_j} can be endowed with the complex structure $\frac{1}{\mu_j^2} (\Lambda_X)^2\Big|_{E_{\mu_j}}$ with induced orientation. We orient $E_0 = T\Sigma$ so that the orientation agrees with the induced orientation from TM. We set the real fiber dimension of E_j to be $2m_j$, so that $\sum_{j=0}^{\tau} m_j = m = \frac{1}{2}\dim M$. We now introduce forms a_j; in the case where E_0,..., E_τ are direct sums of line bundles, they are the first Chern forms (or, classes if considered as elements of $H^*(\Sigma)$) of the line bundle components. In general, let $a_1, ..., a_m$ be such that

(1) The i^{th} Pontryagin class of E_0 is the i^{th} symmetric function of $a_1^2, ..., a_{m_0}^2$, and its Euler class is $a_1 ... a_{m_0}$.
(2) For $i = 1, ..., \tau$, the k^{th} Chern class of E_i the k^{th} elementary symmetric function of those a_j^2 such that $m_0 + ... + m_{i-1} + 1 \leq j \leq m_0 + ... + m_i$.

Let $\lambda_1, ..., \lambda_m$ be the list of real numbers $\underbrace{0, ..., 0}_{m_0 \text{ times}}, \underbrace{\mu_1, ..., \mu_1}_{m_1 \text{ times}}, ..., \underbrace{\mu_\tau, ..., \mu_\tau}_{m_\tau \text{ times}}$, so that they are the skeigen-values of Λ_X. We define

$$\psi(\Lambda_X) := \widehat{\psi}(\lambda_1 + a_1, ..., \lambda_m + a_m).$$

One specific example we will use is

$$\chi(\Lambda_X^\nu) = (\lambda_{m_0+1} + a_{m_0+1}) ... (\lambda_m + a_m).$$

There is a technical change we need to make in the case Σ is a point, in which case $TM = E_1 \oplus ... \oplus E_\tau$, and it may be the case that the orientation induced from the complex structures on the E_j does not produce the given orientation of TM. In this case, we instead let

$$\psi(\Lambda_X) := \widehat{\psi}(-\lambda_1 - a_1, \lambda_2 + a_2, ..., \lambda_m + a_m).$$

Theorem 8.1. *Let (M, g) be a compact, oriented Riemannian manifold of dimension $2m$ that is endowed with an oriented singular Riemannian foliation $\mathcal{F}$. Let X be a nondegenerate transverse Killing vector field on M whose span is $T\mathcal{F}$. Let ϕ be an* $\mathrm{ad}(\mathrm{SO}(2m))$*-invariant symmetric form of degree m on $\mathfrak{o}(2m)$. Then the characteristic number $\phi(M)$ defined by ϕ satisfies*

$$\phi(M) = \sum_j \frac{\phi(\Lambda_X)}{\chi(\Lambda_X^\nu)} [\Sigma_j],$$

where Σ_j are the connected components of the singular stratum Σ of $\mathcal{F}$.

Proof. The vector field X globally generates a singular Riemannian flow. For p near Σ, we replace X with $\widetilde{X} = \frac{d}{dt} \exp^\perp \left(\exp(t\Lambda_X^\nu) \left(\exp^\perp\right)^{-1}(p) \right)\Big|_{t=0}$ where $\exp^\perp : N\Sigma \to M$ is the normal exponential map and $\exp : \mathfrak{so}(2k) \to SO(2k)$ is the Lie group exponential with $2k = 2m - \dim\Sigma$. The flow of this vector field is the same as the flow of X, and in a nice metric, $\widetilde{X}$ is an isometric flow near Σ. We then need only calculate each

$$-\lim_{\delta \to 0} \int_{\partial T_\delta \Sigma_j} \eta_1 \ .$$

We refer to [7, proof of Theorem C] for the calculation of the residue, where the calculation is local and only uses the fact $\widetilde{X}$ is Killing in the small tubular neighborhood. Since the final formula of the limit is the same for both X and $\widetilde{X}$, the result follows. $\square$

Remark 8.2. *For the special case where Σ_j is an isolated fixed point p,*

$$\begin{aligned}\frac{\phi\left(\Lambda_X\right)}{\chi\left(\Lambda_X^{\nu}\right)}\left[\Sigma_j\right] &= \frac{\phi\left(\Lambda_X\right)}{\chi\left(\Lambda_X^{\nu}\right)}(p) \\ &= \frac{\widehat{\psi}\left(\lambda_1+a_1,\lambda_2+a_2,...,\lambda_m+a_m\right)}{\left(\lambda_1+a_1\right)\left(\lambda_2+a_2\right)...\left(\lambda_m+a_m\right)}(p).\end{aligned}$$

Remark 8.3. *The theorem above can easily be adapted to the case where the characteristic numbers come from the curvature of a more general foliated vector bundle over M. In this case, X acts canonically on such a bundle.*

Example 8.4. *The following singular foliation is from* [24, Section 3.4]*. Consider the foliation on S^4 defined as follows. Let $v=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ be an eigenvector of a symmetric matrix $B\in SL\left(2,\mathbb{Z}\right)$ with positive irrational eigenvalues. We consider S^4 to be a suspension of $S^3\subseteq\mathbb{C}^2$, and we foliate each S^3 by the curves $t\mapsto\left(\exp\left(it\alpha\right)z_1,\exp\left(it\beta\right)z_2\right)$. This nonsingular isometric flow on S^3 extends to an isometric flow of S^4, with two fixed points at the poles. Note that each generic leaf closure of the flow is a two-dimensional torus. A tubular neighborhood of such a torus is isometric to a solid torus of the form $D^2\times T^2$, where D^2 is a two-dimensional disk, and where the boundary of this tube is a (rectangular) 3-torus $S^1\times T^2$. Choose two tubes Tube_1 and Tube_2 like this inside S^4 that are isometric and disjoint. We glue the two boundary components of $S^4\setminus\{\mathrm{Tube}_1\cup\mathrm{Tube}_2\}$ via the 3×3 matrix $\begin{pmatrix}1&0\\0&B\end{pmatrix}$, which is a foliated diffeomorphism between the boundary components. This is equivalent to attaching a handle. The result is the manifold*

$$M=\left\{S^4\setminus\{\mathrm{Tube}_1\cup\mathrm{Tube}_2\}\right\}/\sim,$$

where the equivalence relation $\sim$ is given by the gluing map described above. For a small interval I, we use the product metric on $\left(\partial\mathrm{Tube}_1\right)\times I\cong\left(\partial\mathrm{Tube}_2\right)\times I$, and using a basic partition of unity (a partition of unity that is constant on the leaves) we patch this to the original metric on $S^4\setminus\{\mathrm{Tube}_1\cup\mathrm{Tube}_2\}$. The original foliation induces a singular Riemannian flow on M with this metric. It was shown in [24, Section 3.4] *that this flow is not isometric. In fact, we certainly could attach more handles as desired. In any case, we could now compute for example the Euler characteristic of this manifold using Theorem 8.1. The residue at each pole is 1, so that*

$$\chi\left(M\right)=1+1=2.$$

The same result could be obtained from the Hopf index theorem.

Example 8.5. *Consider the manifold $\mathbb{C}P^m$, with homogeneous coordinates $[z_0,...,z_m]$. Consider the isometric flow parametrized by the curves $t\mapsto[z_0,\exp\left(it\alpha_1\right)z_1,...,\exp\left(it\alpha_m\right)z_m]$, where $\left(\alpha_1,...,\alpha_m\right)$ is an eigenvector of a specific matrix $A\in SL\left(m,\mathbb{Z}\right)$, where $\{\alpha_1,...,\alpha_m\}$ is linearly independent over Q. This isometric flow has $m+1$ fixed points $[1,0,...,0]$, $[0,1,0,...,0]$, ...,$[0,0,...,0,1]$. Similar to the last example, we note that generic leaf closures* *of the flow are m-dimensional tori. A tubular neighborhood of such a torus is isometric to a tube of the form $D^m \times T^m$, where D^m is a m-dimensional disk, and where the boundary of this tube is of the form $S^{m-1} \times T^m$. Choose two tubes* Tube_1 *and* Tube_2 *like this inside $\mathbb{C}P^m$ that are isometric and disjoint. We glue the two boundary components of $\mathbb{C}P^m \setminus \{\mathrm{Tube}_1 \cup \mathrm{Tube}_2\}$ via the map* $\mathrm{id}\times A$, *which is a foliated diffeomorphism between the boundary components. The result is the manifold*

$$M = \{\mathbb{C}P^m \setminus \{\mathrm{Tube}_1 \cup \mathrm{Tube}_2\}\} / \sim ,$$

where the equivalence relation $\sim$ is given by the gluing map described above. For a small interval I, we use the product metric on $(\partial\mathrm{Tube}_1) \times I \cong (\partial\mathrm{Tube}_2) \times I$, and using a basic partition of unity we patch this to the original metric on $\mathbb{C}P^m \setminus \{\mathrm{Tube}_1 \cup \mathrm{Tube}_2\}$. The original foliation induces a singular Riemannian flow on M with this metric. Similar to what is shown in [24, Section 3.4], *we can see that this flow is not isometric. However, we may apply Theorem 8.1 to compute the signature of M. On a small neighborhood of each singular point $[0, \dots, z_j = 1, 0, \dots, 0]$, the foliation has the form of the flow*

$$(z_0, z_1, \dots, \widehat{z_j}, \dots, z_m) \mapsto (\exp(-it\alpha_j)z_0, \exp(it(\alpha_1 - \alpha_j))z_1, \dots, \widehat{z_j}, \dots, \exp(it(\alpha_m - \alpha_j))z_m),$$

letting $\alpha_0 = 0$. Then the residue calculation for the signature gives

$$\sigma(M) = \sum_{j=0}^{m} \prod_{i=0,\neq j}^{m} \mathrm{sgn}\,(\alpha_i - \alpha_j) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd} \end{cases} .$$

We see that the surgery did not alter the signature.

9. Generalization of the Álvarez class

Recall that for a Riemannian foliation, the basic component κ_b of the mean curvature one-form is always closed and determines a class in basic cohomology $H_b^1(M, \mathcal{F})$. This class is trivial if and only if the foliation is taut (minimalizable). The form κ_b plays a crucial role in the study of Laplacians and Dirac operators on Riemannian foliations.

In [25], the authors generalize this result to the case of a singular Riemannian foliation $\mathbb{K}$ on a compact manifold X. In the singular case, no bundle-like metric on X can make all the leaves of $\mathbb{K}$ minimal. They prove that the Álvarez classes of the strata can be glued in a unique global Álvarez class. As a corollary, if X is simply connected, then the restriction of $\mathbb{K}$ to each stratum is geometrically taut, thus generalizing a celebrated result of E. Ghys for the regular case.

Theorem 9.1. *(in* [25]*) Let $\mathcal{K}$ be an SRF on a closed manifold X. Then there exists a unique class $\kappa_X \in H_b^1(X, \mathcal{K})$ that contains the Álvarez class of each stratum. More precisely, the restriction of κ_X to each stratum S is the Álvarez class of $(S, \mathcal{K}_S)$.*

10. Differential Operators

There are many well-known properties of elliptic operators such as Laplacians and Dirac operators on manifolds. There are important generalizations of these operators to the equivariant setting (where a compact Lie group acts by isometries) and to the Riemannian foliation setting. Here are some open questions concerning differential operators on SRFs.

Problem 10.1. *Suppose that we are given a smooth basic function f on an SRF, and then use that an initial value for the ordinary heat equation:*

$$\left(\frac{\partial}{\partial t}+\Delta_x\right)u\left(x,t\right)=0;\ \ u\left(x,0\right)=f\left(x\right).$$

Under what conditions is it true that the temperature distribution $u\left(x,t\right)$ is basic for each t? Note that in the case of a compact Lie group action by isometries, this is always true. For Riemannian foliations, this is true exactly when the mean curvature is basic. We conjecture that the same is true for SRFs.

Problem 10.2. *Is there an appropriate Laplacian on basic functions and basic forms of an SRF that has discrete spectrum? If so, does the heat kernel exist and have asymptotics that depend on the geometry? Is basic cohomology finite-dimensional, and is there an appropriate Hodge theorem? Does elliptic regularity hold?*

Problem 10.3. *Are there Dirac operators such as the basic and transverse Dirac operators that are natural for this kind of geometric situation? If so, is the spectrum discrete, and in what ways are they similar to Dirac operators on manifolds and singular spaces? Is there a version of index theory for this situation?*

Problem 10.4. *In the last two problems, does mean curvature play a role as in the case of Riemannian foliations?*

One possible approach to such problems is to work on the principal stratum, where the SRF is a regular Riemannian foliation. This stratum is an open and dense subset of the manifold; however it is noncompact and not complete.

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