THE SELBERG TRACE FORMULA OF COMPACT RIEMANN SURFACES

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1. Introduction to the Selberg Trace Formula

This is a talk about the paper H. P. McKean: *Selberg's Trace Formula as applied to a compact Riemann surface* (1972). For simplicity, assume that M is a compact Riemannian manifold. Consider classical mechanics on M, where a free particle on M moves along geodesics. If M has infinite fundamental group, then in each free homotopy class of a curve on M, there is a unique closed geodesic. If M is a Riemann surface of genus ≥ 1, we can look at the length of the (unique) closed shortest geodesic in each equivalence class from $\pi_1(M)$.

Next, consider quantum mechanics. Eigenvalues of the Laplacian on the manifold

$$0 = \gamma_1 < \gamma_2 \leq \gamma_3 \leq \ldots \uparrow +\infty$$

γ_j is the energy of the j^{th} "pure" state. We expect that there is a relation between the classical data (lengths of closed geodesics) and quantum data (eigenvalues). The Selberg trace formula provides this link. So there should be a "Selberg trace formula" on any manifold. There are many examples of this. When the manifold has a lot of symmetry (eg hyperbolic space mod a subgroup of $PSL(2,\mathbb{R})$), there is an example.

Starting with the 19th century: the **Poisson summation formula**. Let $M = \mathbb{C}\diagup L$ be the 2-torus, where L is an integral lattice. We let L be the integral span of 1 and $a + ib$. Consider the Laplacian on this surface. Then the Laplacian will be

$$\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

acting on L-periodic functions. The eigenfunctions are

$$f(x) = \exp(2\pi i \omega' \cdot x),$$

where ω' is an element of the dual lattice: that is $\omega' \cdot \omega \in \mathbb{Z}$ for all $\omega \in L$. The corresponding eigenvalues are $4\pi^2 |\omega'|^2$. The Poisson summation formula relates the theta functions of L and its dual lattice L', as follows:

$$\sum_{\omega' \in L'} \exp\left(-4\pi^2 \left|\omega'\right|^2 t\right) = \frac{\operatorname{area}(M)}{4\pi t} \sum_{\omega \in L} \exp\left(-\frac{|\omega|^2}{4t}\right),$$

or

$$\underbrace{\sum_{\lambda_j} \exp(-\lambda_j t)}_{\text{quantum side}} = \underbrace{\frac{\operatorname{area}(M)}{4\pi t} \sum_{|\omega| \text{ length of closed geodesics}} \exp\left(-\frac{|\omega|^2}{4t}\right)}_{\text{classical side}}$$

The theta function in physics language is the **partition function**. Also, this is the trace of the heat kernel:

$$\operatorname{tr}(\exp(-t\Delta)) = \sum_{\lambda_j} \exp(-\lambda_j t),$$

where $\exp(-t\Delta)$ is the heat operator. The heat equation is

$$\begin{aligned} \frac{\partial}{\partial t} K &= -\Delta_x K;\ K(0,x,y) = \delta(x-y), \\ \operatorname{tr}(\exp(-t\Delta)) &= \int_M K(t,x,x)\ dx \end{aligned}$$

In the case of the torus,

$$K^T(t,x,x) = \sum_{\gamma \in L} K^{\mathbb{R}^2}(t, \gamma(x), x).$$

On the circle,

$$K^{S^1}(t,x,x) = \sum_{n \in \mathbb{Z}} K^{\mathbb{R}}(t, x+n, x).$$

To prove the Poisson summation formula, one expands the left side in terms of the Fourier series and uses the known heat kernel for $\mathbb{R}$: $K^{\mathbb{R}}(t,x,y) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x-y|^2}{4t}\right)$.

The knowledge of the spectrum of the Laplacian determines $|a|$ and $|b|$, as follows. For our torus with the lattice above, $\operatorname{area}(M) = |b|$. After this, subtract terms from both sides the parts corresponding to geodesics with length $1, 2, 3, \ldots$ The next geodesic will be $\sqrt{|a|^2 + |b|^2}$, so we can find $|a|$. So we can determine the torus up to reflection.

Next, we generalize to a hyperbolic Riemann surface M with genus $g \geq 2$. Then the spectrum $\sigma(M)$ of the Laplacian is

$$0 = \gamma_1 < \gamma_2 \leq \gamma_3 \leq \ldots \to +\infty.$$

The Selberg trace formula relates the trace

$$\mathrm{tr}\left(\exp\left(-t\Delta\right)\right)=\sum_{j=1}^{\infty}\exp\left(-t\lambda_j\right)$$

of the heat kernel to the kind of dual theta function. The role of the lattice L is taken over by the conjugacy classes Q of $G=\pi_1(M)$, identified with a subgroup of $SL(2,\mathbb{R})$. Then

$$M=SL(2,\mathbb{R})\diagup G.$$

The numbers $|w|$ are replaced by

$$l(Q)=2\cosh^{-1}\left(\frac{1}{2}\mathrm{tr}(Q)\right).$$

Here, Q is a free deformation class of closed paths on M, and $l(Q)$ is the length of the shortest path in this class. There is a famous (noncompact) cases that we will not cover

$$M=SL(2,\mathbb{R})\diagup SL(2,\mathbb{Z})$$

or

$$M=SL(2,\mathbb{R})\diagup\Gamma$$

where Γ is an algebraic subgroup of $SL(2,\mathbb{Z})$. Audrey Terras and Serge Lang have good books on the subject. Also there is a survey paper by Werner Müller.

2. Riemann surface formula

Let M be a compact Riemann surface of genus $g\geq 2$. By the Riemann uniformization theorem, the universal cover is the upper half plane $\mathbb{H}$.

$$\mathbb{H}=\{(x_1+ix_2):x_2>0\}$$

This is also called the Poincaré hyperbolic plane, with metric

$$ds^2=\frac{dx_1^2+dx_2^2}{x_2^2}$$

(You can also realize this as the Poincare disk.) The fundamental group $\pi_1(M)$ acts by deck transformations on $\mathbb{H}$ that are isometries $z\mapsto\frac{az+b}{cz+d}$, such that $\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=1$ with $a,b,c,d\in\mathbb{R}$. So $SL(2,\mathbb{R})$ is the group of isometries of $\mathbb{H}$. Thus, we can identify $\pi_1(M)$ with a subgroup G of $SL(2,\mathbb{R})$. The G has a fundamental region in $\mathbb{H}$ that is a hyperbolic polygon with $4g$ sides.

What must be true about G in order that $\mathbb{H}\diagup G$ is a compact Riemann surface? Note that $SL(2,\mathbb{R}) = KAN$, where $K = \mathrm{SO}(2) = \left\{\begin{pmatrix} \cos(\theta) & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\}$ is the stabilizer of i. The group A is the group of magnifications $\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0\right\}$. The group N is the group of horizontal translations, $N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\right\}$.

Proposition 1. *For any $g \in SL(2,\mathbb{R})$, g is conjugate to*

- *rotation iff* $\mathrm{tr}(g) < 2$ *(elliptic)*
- *magnification iff* $\mathrm{tr}(g) > 2$ *(hyperbolic)*
- *translation iff* $\mathrm{tr}(g) = 2$ *(parabolic)*

The hyperbolic distance $d(x,y)$ satisfies

$$\begin{aligned} d(x,gx) &= d(kx,kgx) \\ &= d\left(kx,\left(kgk^{-1}\right)kx\right), \end{aligned}$$

for all k, g in $SL(2,\mathbb{R})$. So

$$\begin{aligned} \inf_x d(x,gx) &= \inf_x d\left(kx,\left(kgk^{-1}\right)kx\right) \\ &= \inf_x d\left(x,\left(kgk^{-1}\right)x\right) \end{aligned}$$

Think of elements of G as homotopy classes of closed paths on M with fixed base point.

Free homotopy classes of M are identified with the conjugacy class $Q = \{kgk^{-1} : k \in G\}$.

Every nontrivial element of G is conjugate to a hyperbolic element. (Proof: if an element g of G is not the identity, then there is a geodesic of minimum length connecting some x to gx, but if g is parabolic or elliptic, this length can go to zero.) Thus it is conjugate to a magnification, and thus $\ell(g^n) = |n|\,\ell(g)$ is true, where $\ell(g) = \inf_x d(x,gx)$ is the length of the shortest path.

For every $g \in G$ that is nontrivial, it can be expressed in a unique way as the positive power of a **primitive** element $p \in G$ (primitive: it is not the power of any other element of G).

Proposition 2. *As p runs through the inconjugate primitive elements in G and n through the positive integers, the conjugacy class*

$$Q = \left\{kp^n k^{-1} : k \in G\diagup G_p\right\}$$

runs through the conjugacy classes of G. Here, G_p is the centralizer of p. Moreover, for fixed p, n, elements kpk^{-1} run once through Q as k runs through $G\diagup G_p$.

Note that $d(x,y) = \cosh^{-1}\left(1 + \frac{\|x-y\|}{2x_2y_2}\right)$, $\ell(p^n) = n\left|\log m^2\right|$ where $p \sim \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$.

Theorem 3. *(Selberg trace formula) Start with a function* $K : \mathbb{R} \to \mathbb{R}$ *that decays sufficiently rapidly as* $x \to \infty$. *Then* $K_H(x,y) = K(\cosh d(x,y))$ *is a function on* $\mathbb{H} \times \mathbb{H}$. *It induces a symmetric kernel on* $M \times M$ *via*

$$K_M(x,y) = \sum_{g \in G} K_H(x, gy).$$

Then

$$K_M(x, hy) = K_M(x,y)$$

for all $h \in G$. *Then, with* $dx = \frac{dx_1 dx_2}{x_2^2}$ = *hyperbolic volume element*

$$\begin{aligned} \mathrm{tr} K_M &:= \int_M K_M(x,x)\ dx \\ &= \mathit{area}(M)\, K(1) + \sum_{n=1}^{\infty} \sum_{\substack{\text{inconjugate} \\ \text{primitive } p}} \frac{\ell(p)}{\sqrt{\cosh \ell(p^n) - 1}} \int_{\cosh \ell(p^n)}^{\infty} \frac{K(b)\, db}{\sqrt{b - \cosh \ell(p^n)}}. \end{aligned}$$

Proof. Let F be a fundamental domain of M. We have

$$\begin{aligned} \mathrm{tr} K_M &= \int_F K_M(x,x)\ dx \\ &= \sum_{g \in G} \int_M K(\cosh d(x, gx))\ dx \\ &= \mathrm{area}(M)\, K(1) + \sum_{n=1}^{\infty} \sum_{\substack{\text{inconjugate} \\ \text{primitive } p}} \sum_{k \in G/G_p} \int_F K\left(\cosh d\left(x, kp^n k^{-1} x\right)\right)\ dx \\ &= \mathrm{area}(M)\, K(1) + \sum_{n=1}^{\infty} \sum_{\substack{\text{inconjugate} \\ \text{primitive } p}} \sum_{k \in G/G_p} \int_F K(\cosh d(x, p^n x))\ dx \\ &= \mathrm{area}(M)\, K(1) + \sum_{n=1}^{\infty} \sum_{\substack{\text{inconjugate} \\ \text{primitive } p}} \int_{F_p} K(\cosh d(x, p^n x))\ dx \end{aligned}$$

where F_p is a fundamental domain for G_p. Then p is conjugate to some magnification $x \mapsto m^2 x$. Then $F_p = \left\{x_1 \in \mathbb{R} : 1 \le x_2 \le m^2\right\}$. The formula follows from a direct calculation. $\square$

In the particular case where $K_H(t)$ is the fundamental solution of the heat equation on $\mathbb{H}$. Then $K_H(t) = \exp(-t\Delta)$. Then

$$\begin{aligned}
\operatorname{tr}(\exp(-t\Delta)) &= \sum_{n=0}^{\infty} \exp(-t\gamma_n) \\
&= \operatorname{area}(M) \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{b e^{-b^2/4t}}{\sinh\left(\frac{1}{2}b\right)} db \\
&\quad + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\substack{\text{inconjugate} \\ \text{primitive } p}} \frac{\ell(p)}{\sinh\left(\frac{1}{2}\ell(p^n)\right)} \frac{e^{-t/4}}{(4\pi t)^{1/2}} e^{-|\ell(p^n)|^2/4t}.
\end{aligned}$$