GAGA seminar
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Divergence in Coxeter Groups

Ignat Soroko, UNT

Joint work with P. Dani, Y. Naqvi, A. Thomas

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Divergence:

X is a 1-ended geodesic metric space

distance function

Ends:

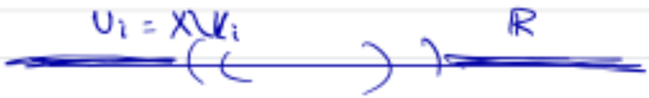
... $K_1 \subset K_2 \subset \ldots$

$\bigcup K_i = X$

Then for all $N >> 0$ $X \setminus K_N$ connected

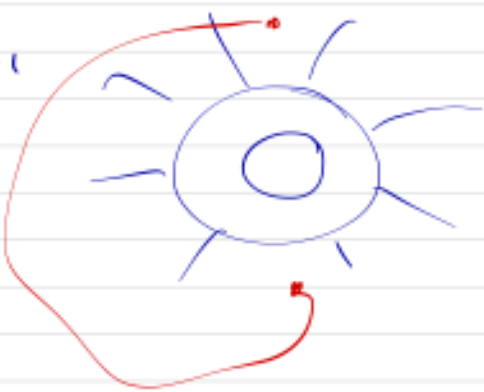
\# conn. components = # ends

An end "$U_1 \supset U_2 \supset \ldots U_n \supset \ldots$" "1 end"



$\mathbb{R}$ has 2 ends

$\mathbb{R}^n$ $n > 1$

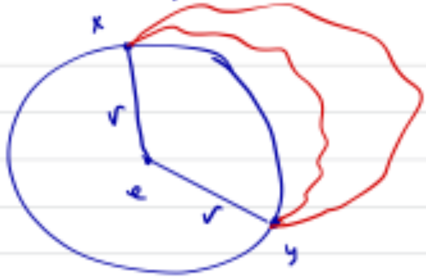


1-ended

infinite tree

∞ many ends

Take a open ball centered at some pt e of radius r



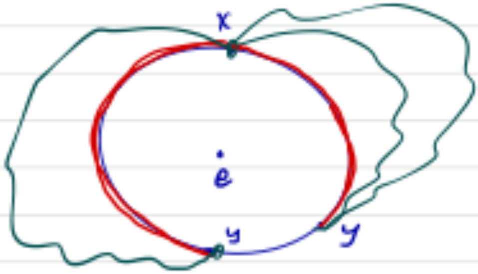
$B(e,r)$

$S(e,r)$ — sphere

r-avoidant path from x to y on $S(e,r)$ lie outside $B(e,r)$.

$$\mathrm{div}_{X,e}(r) = \sup_{x,y \in S(e,r)} \inf \left(\begin{array}{l} \text{length of } r\text{-avoidant} \\ \text{paths from } x \text{ to } y \end{array} \right)$$

X



"worst student in the honors calculus class"

Examples:

$\mathbb{R}^2$:

πr

$\mathrm{div}_{\mathbb{R}^2}(r) \simeq r$ (linear)

$\mathbb{H}^2$:

$\sim O(e^r)$

$= \pi \cdot \sinh(r) \sim \pi \dfrac{e^r - e^{-r}}{2} \simeq e^r$ (exponential)

$\mathrm{div}_{\mathbb{H}^2}(r) \simeq \exp(r)$.

Symmetric spaces of non-compact type:

euclidean div ~ linear

non-positively curved div ~ exp

Gromov '93 : same dichotomy should hold for semi-hyperbolic spaces. div either $\sim r$

$\sim \exp(r)$

Semi-hyperbolic $\supset$ CAT(0) spaces:

" Triangles are not fatter than in $\mathbb{R}^2$ "

CAT(0) X:

3 geodesics

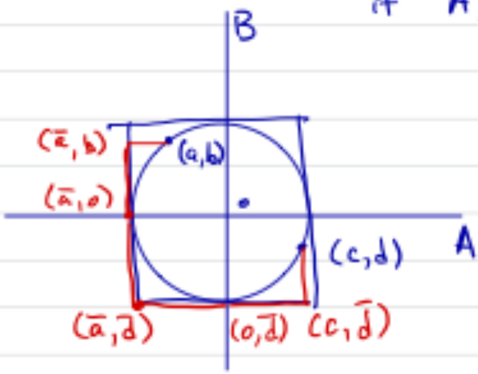
$$\boxed{d_X(p,q) \leq d_{\mathbb{R}^2}(\bar{p},\bar{q})}$$

CAT(0)

CAT(0) a generalization of euclidean and hyperbolic.

Remarkable fact: If $X = A \times B$, then $\mathrm{div}_X \simeq$ linear

if A, B spaces with extendable geodesics



$$\mathrm{div}_{A\times B}(r) \leq 6r$$

Perspective: filling functions: M Riemannian manifold

CW cx

α : 1-loop, 1-chain

filling function (r) =

$$= \sup_{|\alpha| \leq r} \inf_{\mathcal{D}} \{ \mathrm{Area}(\mathcal{D}) \mid \partial\mathcal{D} = \alpha \}$$

div is a filling function for $\alpha = S^0$ (:)

in a neighborhood of ∞

$|\alpha| = r$

Groups as metric spaces : countable abstract gps

Cayley graph : $G = F(\underline{S})/\langle\langle R \rangle\rangle$

S = generators

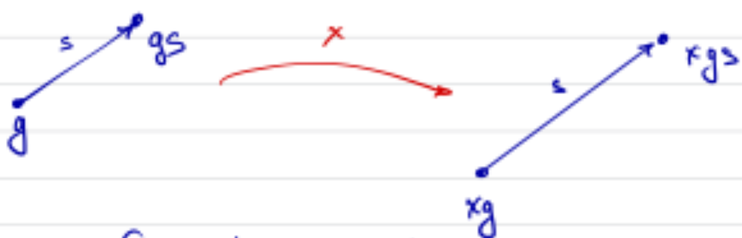
vertices $\longleftrightarrow$ G itself

dir. edges $\longleftrightarrow$ $G \times S$

each edge has length 1

$\Downarrow$

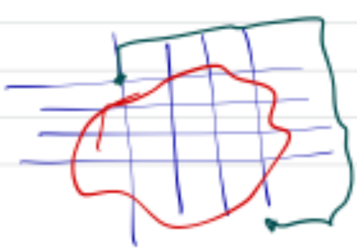
metric space



G acts on Cay by isometries.

Q: Take your favorite class of groups. What divergence functions are possible?

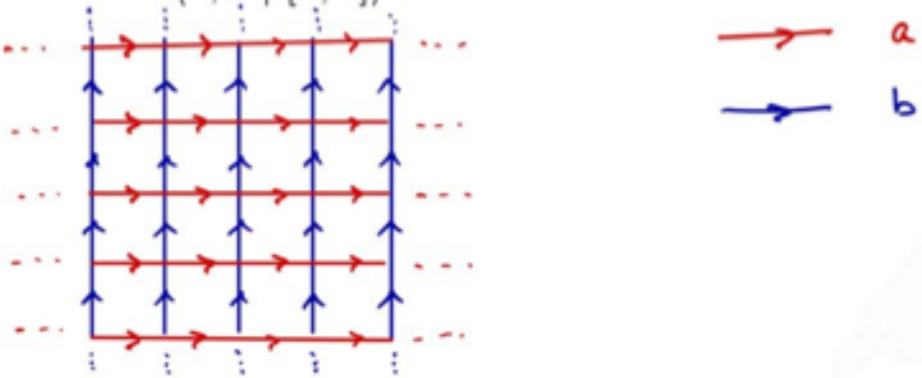
$\mathbb{Z}^2$ $\quad$ $\mathbb{Z}^n$



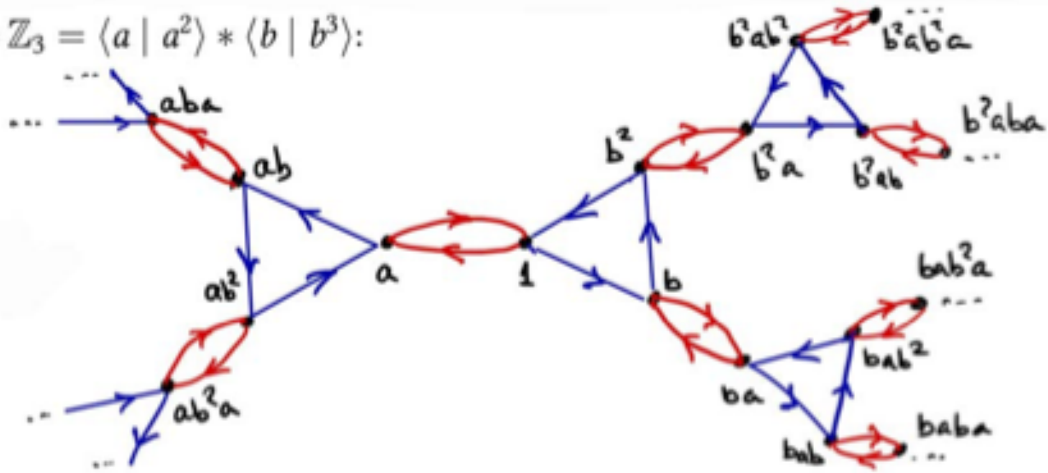
$\boxed{\text{Q (Greg): what is } \mathrm{div}_{\mathbb{Q}}\text{? (}\mathbb{Q}\text{ is infinitely generated, so its Cayley graph is locally infinite)}}$

Examples of Cayley graphs:

$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid [a,b] \rangle$:



$\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a \mid a^2 \rangle * \langle b \mid b^3 \rangle$:



Examples of CAT(0) groups with polynomial div

Gersten '94 $\quad \mathrm{div}_G \sim r^2$

Macura '2003 $\quad \sim r^3$

2013 $\quad \sim r^d, \ d > 1$

Gromov's question has negative answer.

COXETER GROUPS

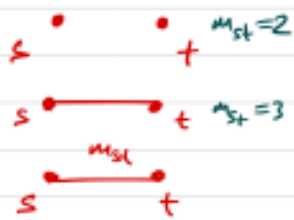
Is given by the data:

S, a finite generating set

$(m_{st})_{s,t \in S}$ of $\{1,2,3,\dots\} \cup \{\infty\}$

- symmetric
- $m_{ss} = 1 \quad \forall s \in S$

Coxeter graph



The Coxeter gp (W, S) is given by:

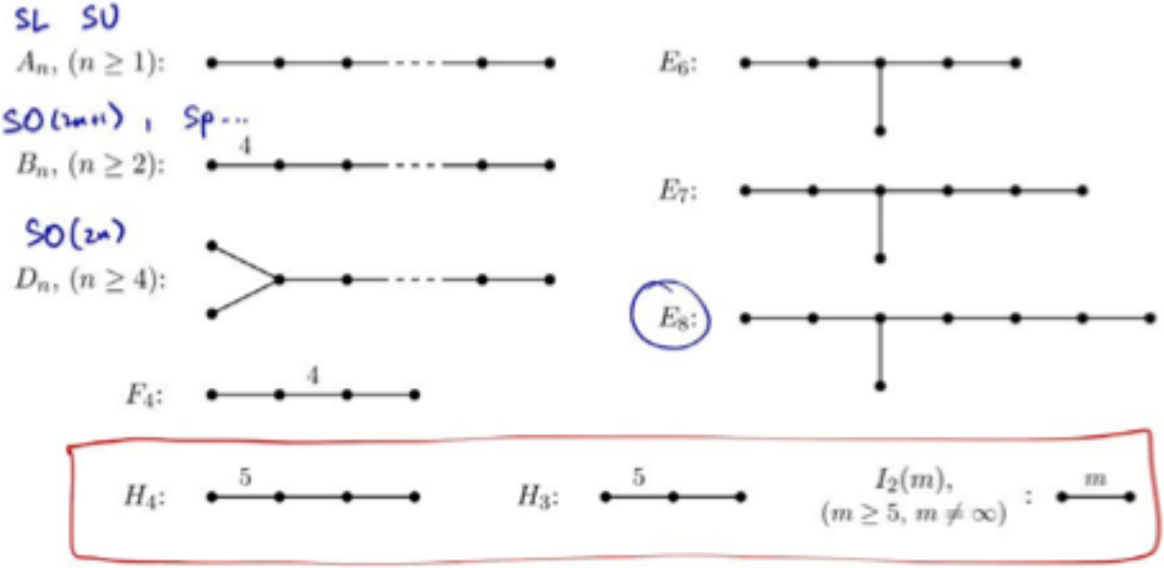
$$W = \langle S \mid (st)^{m_{st}} = 1, \ s,t \in S, \ m_{st} \neq \infty \rangle$$

$ss' = 1$, $s^2 = 1$ each generator $s \in S$ is an involution.

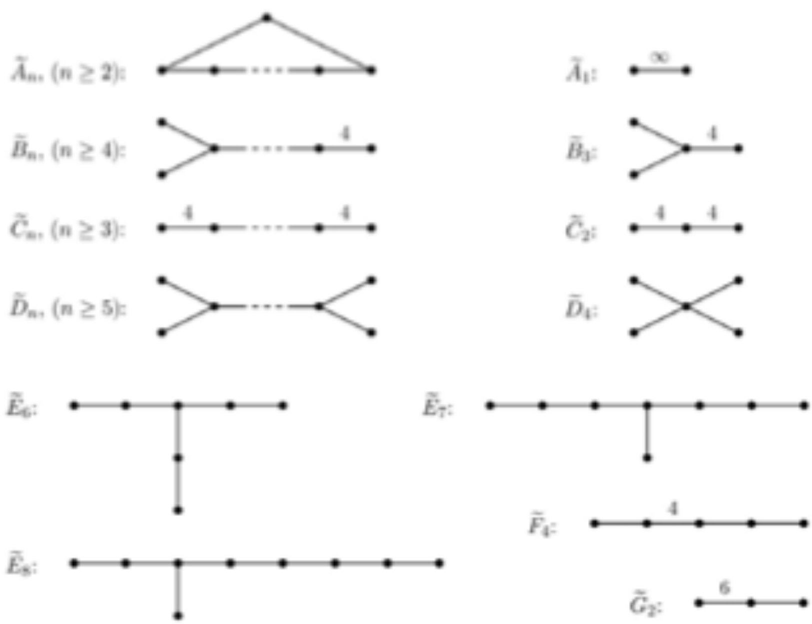
Crystallographic groups, groups gen. by reflections are Coxeter.

① Sperical Coxeter groups $\equiv$ finite

Irreducible ones (i.e. directly indecomposable) are Weyl groups of simple cx Lie algebras, plus additional ones: $H_3, H_4, I_2(m)$, which do not preserve lattice in $\mathbb{R}^n$.
$m \neq 2,3,4,6$



② Affine Coxeter groups ≡ contain $\mathbb{Z}^n$ as a finite index subgroup



they correspond to "extended Dynkin diagrams"

Our results:

Thm 1: If (W,S) is 1-ended, irreducible and non-affine
$\Rightarrow$ $\mathrm{div}_W(r) \succeq r^2$

Cor: characterization of linear divergence:
div_W is linear $\Longleftrightarrow$ $W = W_1 \times W_2$ with

product • both W_1, W_2 infinite, or
• $\underbrace{W_1 \text{ finite}}_{\text{ignored}}$ and W_2 irreducible affine of rank ≥ 3

has finite index subgroup $\mathbb{Z}^{r-1}$ also a product if $r-1 \geq 2$

(Nothing else happens)

Cor: If div_W is superlinear $\Rightarrow$ it's at least quadratic.

There is a gap between r and r^2!

We introduce a combinat. invariant called a hypergraph index, directly computable from the Coxeter graph, which is an integer or ∞

Thm 2: If (W,S) is 1-ended:

① $h=0 \Leftrightarrow \mathrm{div}_W$ is linear

② $h = 1 \Rightarrow$ div is quadratic

③. h is finite $\Rightarrow$ $\mathrm{div}(r) \lesssim r^{h+1}$

④. h is ∞ $\Leftrightarrow$ div is exponential

Conj: h is finite $\Leftrightarrow$ $\mathrm{div}(r) \asymp r^{h+1}$

Levcovitz proved for Right-angled Coxeter gps $m_{st} = 2, \infty$

We proved for some infinite series

Th 3: If h is finite $\Rightarrow$ $h \leq b_1(\Delta) + 1$

b_1 = first Betti number

Δ = Coxeter graph of (W, S)

Cor: If W is 1-ended, and Δ is a tree, then div is linear, quadratic or exponential only, and all of them are realized.

thick edge : $m_{st} = 4$

affine, contains $\mathbb{Z}^7$ as f. index subgp.