AN ENTERTAINING PROOF USING INVERTIBLE COBORDISMS AND AN INFINITE PROCESS TRICK

A paper on singular intersection homology refers to Stallings' *Ingenious Invertible Cobordism trick*.

Notation 0.1. *cX means open cone on X, $\bar{c}X$ is the closed cone on X.*

Theorem 0.2. *Suppose X and Y are compact topological spaces, and suppose there is a neighborhood U of the vertex v of cX such that $(U, v) \cong (cY, v')$. Then $(cX, v) \cong (cY, v')$.*

Note that it is not true that $X \cong Y$. A counterexample is provided by the

Theorem 0.3. *(Double Suspension Theorem, Cannon, Edwards) Suppose M is a homology sphere. Then $\Sigma^2 M \cong S^{n+2}$ (where Σ denotes the suspension).*

(and $S^{n+2} \cong \Sigma^2 S^n$.) So either M or ΣM is possibly not a sphere, but the cone on it is the same as the cone on a sphere.

Corollary 0.4. *There exist triangulations of manifolds that do not exhibit the manifold as a combinatorial manifold. (ie star neighborhood of vertex is not a sphere)*

(Start with simplicial homology sphere with triangulation – suspend it twice.)

Proof. (Proof of first theorem) Draw a picture of a cone. and of U. Can retract cX along cone lines to a smaller cone (using compactness), whose boundary is a copy of X. Now play game again with a cone on Y, and you can find a smaller neighborhood of the cone point whose boundary is a copy of Y. So we have a Y collared cobordant through P to X, collared cobordant through Q to Y, and could keep going. Then $PQ = P \cup_X Q \cong Y \times [0,1]$, $QR = Q \cup_Y R \cong X \times [0,1]$. Also get $RQ = R \cup_X Q \cong Y \times [0,1]$ (using copy of Q). Then $RQ \cong (Y \times [0,1])\, RQ \cong PQRQ = P\,(QR)\,Q = P\,(X \times [0,1])\,Q \cong PQ \cong Y \times [0,1]$.

Then let $N = \bar{c}Y P, M = \bar{c}Y PQ$. Then

$$
\begin{aligned}
& NQRQRQR... \\
= \;& \bar{c}X\,(X \times I)\,(X \times I)\,... = \bar{c}X \\
= \;& (NQ)\,(RQ)\,(RQ)\,... \\
= \;& \bar{c}Y\,(Y \times I)\,(Y \times I)\,.... = \bar{c}Y.
\end{aligned}
$$

□

As an encore:

Theorem 0.5. *(Eilenberg Swindle) If P is a projective R-module (direct summand of a free module), there exists a free module F such that $P \oplus F$ is also free.*

Proof. By assumption, there exists Q such that $P \oplus Q \cong f$ (some free module). Then

$$
\begin{aligned}
& P \oplus Q \oplus P \oplus Q \oplus ... \\
= \;& P \oplus f \oplus f \oplus .... = P \oplus F \\
= \;& (P \oplus Q) \oplus P \oplus Q... \\
= \;& f \oplus f \oplus ... = F.
\end{aligned}
$$

∎

Another encore:

Theorem 0.6. *There exists an irrational number x and an irrational number y such that x^y is rational.*

Proof. If $\sqrt{2}^{\sqrt{2}}$ is rational, we are done ($x = y = \sqrt{2}$). Otherwise, $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$, and then $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$. ∎

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