Definition: A metric space is <u>proper</u> if closed & bounded sets are compact.

Compare this with the definition of a proper map in topology: if $X, Y$ are top spaces, a function $f: X \to Y$ is proper if $f^{-1}(K)$ is compact in $X$ for every compact subset $K$ of $Y$.

A metric space $X$ is proper if & only if $f: X \to \mathbb{R}$ defined by $f(x) = d(x_0, x)$ for some fixed $x_0 \in X$ is a proper function.

---

The following theorem is a generalization of the Hopf-Rinow theorem from differential geometry:

Theorem: Every complete locally compact length space is proper. Conversely, every proper length space is complete and locally compact. Furthermore, all such spaces are geodesic spaces.

---

Definition: Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces. A map $f: X \to Y$ is <u>Lipschitz</u> if $\exists$ a constant $C$ such that

$$d_Y(f(x), f(x')) \leq C d_X(x, x')$$

for all $x, x' \in X$.

---

Important example: Every smooth map between compact Riemannian manifolds is Lipschitz.

Definition: A homeomorphism $\phi : (X, d_X) \to (Y, d_Y)$ such that both $\phi$ and $\phi^{-1}$ are Lipschitz is called a bi-Lipschitz homeomorphism

Theorem (Sullivan): Two smooth manifolds of dimension other than four (!) are bi-Lipschitz homeomorphic if & only if they are (topologically) homeomorphic

Counterexamples in dimension 4.

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Definition: Take $f : (X, d_X) \to (Y, d_Y)$, $x_0 \in X$, $a > 0$. The distortion of $f$ at $x_0$ is defined to be

$$D_f(x_0; a) = \frac{\sup\{d_Y(f(x_0), f(x)) : d_X(x_0, x) = a\}}{\inf\{d_Y(f(x_0), f(x)) : d_X(x_0, x) = a\}}$$

Roughly speaking, $D_f(x_0; a)$ is a measure of how far the image $f(S(x_0; a))$ of the sphere of radius $a$ centered at $x_0$ deviates from being a sphere in $Y$ centered at $f(x_0)$.

Definition: We say $f: X \to Y$ is <u>K-quasiconformal</u>

if there exists a constant $K$ such that

$$\limsup_{a \to 0} D_f(x;a) \leq K$$

for all $x \in X$. We say $f$ is ~~quasi-co~~ <u>quasiconformal</u> if it is $K$-quasiconformal for some $K$.

Examples: Every bi-Lipschitz homeomorphism is quasiconformal.

- A conformal map in the sense of Riemannian geometry is 1-quasiconformal.

## Section 1.3

Recall the following theorem from point-set topology:

Theorem: Let $(X,d)$ be a metric space, & define $\tilde{d}: X \times X \to [0,1]$ by $\tilde{d}(x,y) = \min\{d(x,y), 1\}$. Then $\tilde{d}$ is a metric on $X$, and $(X,d)$ and $(X,\tilde{d})$ are homeomorphic.

---

In other words, the topology of $(X,d)$ only depends on the <u>small-scale</u> structure of $d$. In coarse geometry, we consider the <u>large-scale</u> structure of $d$.

Definitions: Let $X, Y$ be metric spaces & let $f: X \to Y$ be a function, <u>not necessarily continuous</u>.

(a): $f$ is (metrically) <u>proper</u> if $f^{-1}(B)$ is a bounded subset of $X$ for every bounded subset $B$ of $Y$.

(b) $f$ is (uniformly) <u>bornologous</u> if for every $R > 0$ $\exists\, S > 0$ such that

$$d_X(x, \tilde{x}) < R \implies d_Y(f(x), f(\tilde{x})) < S.$$

(c) $f$ is <u>coarse</u> if it is both proper and ~~bor~~ bornologous.

Remarks:

1. Metric properness is not generally the same as topological properness, but the two concepts coincide if $X, Y$ are proper metric spaces and $f$ is continuous. Henceforth we shall use "proper" to mean "metrically proper."

2. Note that the definition of bornologous is sort of the converse to the definition of uniform continuity!

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Examples with $X = Y = \mathbb{N}$.

① $f(n) = 14n + 78$ coarse

② $g(n) = 1$ not proper

③ $h(n) = n^2$ not bornologous.

Definition: Let $X, Y$ be metric spaces. We say $f: X \to Y$ is *large-scale Lipschitz* if there exist positive constants $c, A$ such that

$$d(f(x), f(\tilde{x})) \leq c \cdot d(x, \tilde{x}) + A$$

for all $x, \tilde{x} \in X$.

---

A large-scale Lipschitz ~~map~~ map is ~~be~~ bornologous, but the converse is not generally true. However, we do have the following result:

Proposition: Let $X$ be a length space and $Y$ any metric space, & let $f: X \to Y$ be a map. TFAE:

(a) $f$ is large-scale Lipschitz;

(b) $f$ is bornologous;

(c) there exist $R, S > 0$ such that

$$d(x, \tilde{x}) < R \Rightarrow d(f(x), f(\tilde{x})) < S$$

for all $x, \tilde{x} \in X$.

**Definition:** Maps $f, \tilde{f} : X \to Y$ are <u>close</u> if there exists a constant $M > 0$ such that

$$d(f(x), f(\tilde{x})) \leq M$$

for all $x, \tilde{x} \in X$.

---

**Proposition:** $f, \tilde{F}$ are close if & only if there exists a coarse map

$$F : X \times [0,1] \to Y$$

such that $F(x,0) = f(x)$, $F(x,1) = \tilde{f}(x)$ $\forall x \in X$.

---

For this reason, "close" is sometimes called "bornotopic".

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**Definition:** Metric spaces $X, Y$ are <u>coarsely equivalent</u> if there exist coarse maps $f : X \to Y$, $g : Y \to X$ such that $g \circ f$ is close to $id_X$ and $f \circ g$ is close to $id_Y$.

**Remark:** Coarse equivalence is also called "bornotopy equivalence."

Proposition: $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent.

Proof: Let $f: \mathbb{Z} \to \mathbb{R}$ be the inclusion map and define $g: \mathbb{R} \to \mathbb{Z}$ by $g(x) = \lfloor x \rfloor$. Then $g \circ f = id_{\mathbb{Z}}$, and $f \circ g(x) = \lfloor x \rfloor$ is close to $id_{\mathbb{R}}$: in fact, we can take $M = 1$.

Remark: We will <u>see</u> later that $\mathbb{R}$ + $\mathbb{R}^2$ are <u>not</u> coarsely equivalent.

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Next time

Let $\Gamma$ be a discrete group, + let $S$ be a set of generators of $\Gamma$. For each $\gamma \in \Gamma$, let $|\gamma|$ denote the smallest integer $n$ such that

$$\gamma = s_1 s_2 \dots s_n, \quad s_i \in S \text{ or } s_i^{-1} \in S \quad \forall i.$$

It is easy to see that

$$|\gamma \tau| \leq |\gamma| + |\tau|,$$