Examples:

1. $\mathcal{A} = \{f \in C_b(X) : f - \lambda \in C_0(X) \text{ for some } \lambda \in \mathbb{C}\}$
   $Y = X^+$, the one-point compactification of X

2. $\mathcal{A} = C_b(X)$
   $Y = \beta X$, the Stone-Čech compactification of X

3. $X = \mathbb{R}$, $\mathcal{A} = \{f : \mathbb{R} \to \mathbb{C},\ \lim_{x\to-\infty} f(x),\ \lim_{x\to\infty} f(x) \text{ exist}\}$
   $Y = [-\infty, \infty]$

Next time:

Theorem: Let X be paracompact and locally compact Hausdorff, & take $E \subset X \times X$. Let $\overline{X}$ be a compactification of X, & let $\partial X := \overline{X} \setminus X$.

TFAE:

(a) the closure $\overline{E}$ of E in $\overline{X} \times \overline{X}$ intersects the complement of $X \times X$ only in the diagonal $\Delta_{\partial X} = \{(w,w) : w \in \partial X\}$

(b) E is proper (i.e., $E[K]$, $E^{-1}[K]$ are relatively compact when K is relatively compact), and for any net $\{(x_\alpha, y_\alpha)\}$ in E, if $\lim_\alpha x_\alpha = w$, then $\lim_\alpha y_\alpha = w$ as well

(c) E is proper, and for every $w \in \partial X$ and every nbhd V of w in $\overline{X}$, there is a nbhd $U \subseteq V$ of w in $\overline{X}$ with the property that

$$E \cap \left(U \times (X \setminus V)\right) = \emptyset.$$

Furthermore, the collection of sets E satisfying these equivalent conditions are the entourages for a proper connected coarse structure on X.

Definition: This coarse structure on X is the <u>topological coarse structure</u> or the <u>continuously controlled coarse structure</u> on X associated to the compactification $\bar{X}$.

Examples:

① X^+: $\mathcal{E}$ = collection of all proper subsets of $X \times X$
(indiscrete coarse structure)

② βX: $\mathcal{E}$ = collection of all subsets of $X \times X$ with only finitely many points off the diagonal in $X \times X$
(discrete coarse structure)

Proposition: Let X, Y be locally compact Hausdorff spaces with <u>second countable</u> compactifications $\overline{X}, \overline{Y}$. A continuous and proper map $f: X \to Y$ is coarse (w.r.t. $\overline{X}, \overline{Y}$) if + only if f extends to a continuous map $\overline{f}: \overline{X} \to \overline{Y}$.

Remark: If $\overline{f}: \overline{X} \to \overline{Y}$ exists, then $f: X \to Y$ is coarse without the hypothesis of second countability of $\overline{X}, \overline{Y}$, but the converse requires this hypothesis.

Question: Suppose $\mathcal{E}$ is a coarse structure on a locally compact paracompact Hausdorff space X. Under what conditions ~~does~~ ~~$\mathcal{E}$~~ is $\mathcal{E}$ the topological coarse structure associated to a compactification of X?

Partial Answer: Recall that a coarse structure on a paracompact Hausdorff space X is <u>proper</u> if

(i) there is a controlled nbhd of the diagonal;

(ii) every bounded subset B of X has compact closure. (i.e., $B \times B$ is an entourage)

(X is necessarily locally compact)

Let X be a paracompact Hausdorff space equipped with a proper coarse structure. For each $f: X \to \mathbb{C}$ that is bounded and continuous, define $df: X \times X \to \mathbb{C}$ by the formula

$$df(x,y) = f(x) - f(y).$$

We say f is a <u>Higson function</u> if for each entourage E, the restriction of df to E vanishes at infinity.

Proposition: The Higson functions on a proper coarse space form a unital C^*-subalgebra of $C_b(X)$

Proof: The only nonobvious point to check is closure under multiplication, which follows easily from the identity

$$d(fg)(x,y) = df(x,y)\,g(x) + f(y)\,dg(x,y).$$

We let $C_h(X)$ denote the collection of Higson functions on X, & we denote the corresponding compactification by hX. This is the <u>Higson compactification</u> ~~of the~~ associated to the proper coarse structure on X, & the set $hX \setminus X$ is called the <u>Higson corona</u>; it is ~~denoted~~ denoted νX.

Definition: Let X be a proper coarse space. A <u>coarse compactification</u> of X is a compactification whose top. coarse structure is coarser (i.e., has more entourages) than the original coarse structure on X.

Example: The one-point compactification of X is always a coarse compactification.

Proposition: The Higson compactification hX of X is a ~~coarse~~ coarse compactification that is universal in the following sense: given a coarse compactification $\overline{X}$ of X, the identity map $i: X \to X$ extends uniquely to a continuous surjection from hX to $\overline{X}$.

The Higson *compactification* hX can only be defined when X is a proper coarse space. However, the Higson *corona* νX can be defined for *any* coarse space. Here's how.

Definition: Let X be a coarse space & $f: X \to \mathbb{C}$ a function. We say f *tends to 0 at infinity* if for every $\varepsilon > 0$ there exists a bounded set B in X such that $|f(x)| < \varepsilon$ for $x \in X \setminus B$. We say a function $\phi: X \times X \to \mathbb{C}$ *tends to 0 at infinity* if for every $\varepsilon > 0$ there exists an entourage E such $|\phi(x,y)| < \varepsilon$ for all $(x,y) \in (X \times X) \setminus E$.

Let

$B_h(X) = \{ f : X \to \mathbb{C} \text{ bounded} : df \text{ tends to } 0 \text{ at } \infty \}$

$B_0(X) = \{ f : X \to \mathbb{C} \text{ bounded} : f \text{ tends to } 0 \text{ at } \infty \}$

Then $B_0(X)$ is an ideal in $B_h(X)$, and

- $C_0(X) = C_h(X) \cap B_0(X)$;
- $B_h(X) = C_h(X) + B_0(X)$.

Now, if X is a proper coarse space, then

$$C(\nu X) = \frac{C_h(X)}{C_0(X)}.$$

But by the 2nd Isomorphism Theorem from algebra,

$$\frac{C_h(X)}{C_0(X)} = \frac{C_h(X)}{B_0(X) \cap C_h(X)} = \frac{B_0(X) + C_h(X)}{B_0(X)} = \frac{B_h(X)}{B_0(X)}.$$

Thus we can define νX as the maximal ideal space of the commutative C^*-algebra $B_h(X) / B_0(X)$.

Proposition: Let X, Y be coarse spaces. A coarse map $\phi: X \to Y$ extends to a continuous map $\nu\phi: \nu X \to \nu Y$. Moreover, if ϕ and ψ are close, then $\nu\phi = \nu\psi$.

Corollary: If X, Y are coarsely equivalent, then νX and νY are homeomorphic.

Note that we have two constructions here.

- The Higson construction takes a proper coarse structure + associates to it a compactification.
- The continuous control construction takes a compactification + associates to it a coarse structure.

~~Let "h" denote the first construction + "t" the second~~

Question: To what extent are these inverses to one another?

Answer: These are not inverses in general, but we do have the following two results:

Proposition: Let (X, d) be a proper metric space. Then the bounded coarse structure on X is the topological coarse structure assoc. to its Higson compactification.

Proposition: Suppose X is a locally compact Hausdorff space that is equipped with the top. coarse structure assoc. to a 2nd countable compactification $\overline{X}$. Then the Higson compactification hX of X coincides with $\overline{X}$.