Theorems Proved Using Ideas From Coarse Geometry

Definition: ~~A compact~~ Riemannian manifold is __hyperbolic__ if it has constant sectional curvature -1.

M hyperbolic $\Leftrightarrow$ $\tilde{M}$ is isometric to $\mathbb{H}^n$ for some n.

Theorem (Mostow): Let M, N be compact hyperbolic manifolds of dimension $n \geq 3$. If M & N are homotopy equivalent, then they are isometric. Moreover, if $f: M \to N$ is a homotopy equivalence, then it is homotopic to an isometry.

Remark: This result is very false for $n = 2$.

Definition: Let X be a coarse space. The __quasi-isometry group__ $Qis(X)$ is the group of closeness classes of coarse equivalences from X to X.

Recall: X, Y metric spaces, $f: X \to Y$ a map, $x \in X$, $a > 0$. Define

$$D_f(x:a) = \frac{\sup\{d(f(x), f(x')) : d(x,x') = a\}}{\inf\{d(f(x), f(x')) : d(x,x') = a\}}.$$

Suppose $\exists$ a constant K such that

$$\limsup_{a \to 0} D_f(x;a) \le K \quad \forall x \in X.$$

Then we say f is <u>K-quasiconformal</u>, + in general say f is <u>quasiconformal</u> if it is K-quasiconformal for some K.

Key result in proving Mostow Rigidity:

Theorem: For $n \geq 3$, every coarse equivalence $\mathbb{H}^n \to \mathbb{H}^n$ extends by continuity to a homeomorphism $S^{n-1} \to S^{n-1}$ from the ideal boundary of $\mathbb{H}^n$ to itself. This homeomorphism is quasiconformal, + the process of extension to the boundary determines an isomorphism from $QIs(\mathbb{H}^n)$ to the group of quasiconformal homeomorphisms from S^{n-1} to itself.