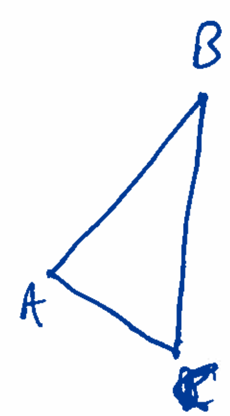
Chapter I:

Parallel Postulate: given a point P and a line ℓ not containing P, there exists a unique line containing P that does not intersect ℓ.

Equivalent formulation: the sum of the angles of a triangle is π.

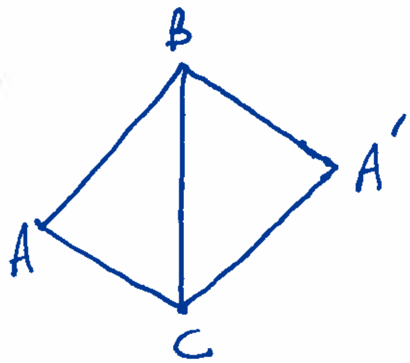
"Theorem" (Legendre): The sum of the angles of a triangle cannot be less than π.

"Proof": Suppose ABC is a triangle whose angle sum is less than π:



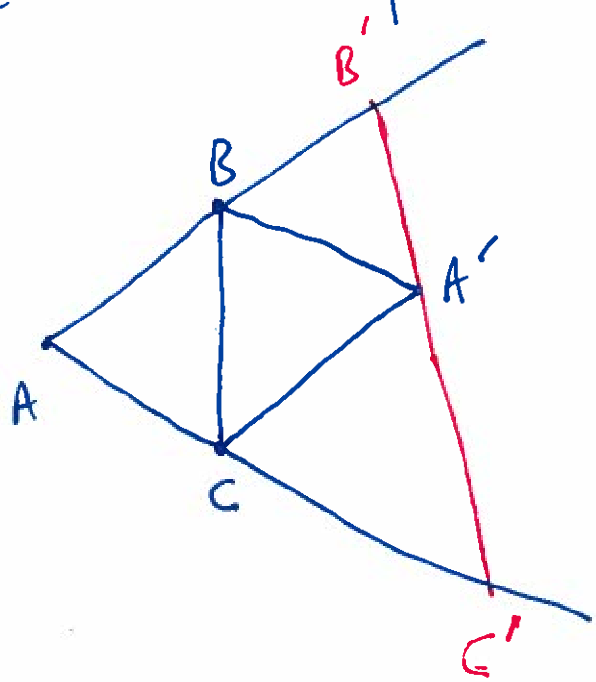
Let δ be its defect; i.e., the amount the angle sum is less than π

Construct another triangle CBA' by rotating ABC through π about the midpoint of BC:



(opposite sides look parallel!)

Next, draw any line through A' that meets the lines ~~segments~~ determined by AB and AC; let B' and C' denote the points of intersection

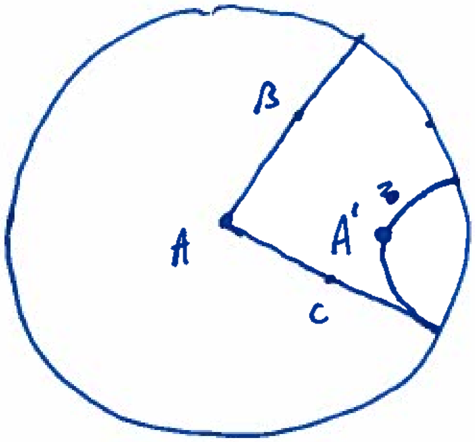


Consider the triangle $AB'C'$. It is made up of four sub-triangles. Two of the triangles; namely ABC and $A'BC$, are congruent, & therefore have the same defect δ. The other two triangles have unknown defect, but their defects are at least nonnegative.

It is easy to show that defect is additive, so we conclude that $AB'C'$ has defect at least 2δ. Iterating this process, we see that we can construct triangles whose defect is arbitrarily large. But obviously no triangle can have defect greater than π, so we have derived our contradiction. . ." $\blacksquare$

Problem: There may not exist a line through A' that intersects both the line containing AB + the line containing AC

Consider the disk model of the hyperbolic plane:



Even though Legendre's proof is flawed, it contains an important perspective: instead of looking at small triangles + their defects (differential geometry), he instead looked at large triangles + what happens as they get larger. This is the idea coarse geometry seeks to capture.

Metric spaces and length spaces

Suppose X is a metric space with metric (distance function) $d: X \times X \to [0,\infty)$.

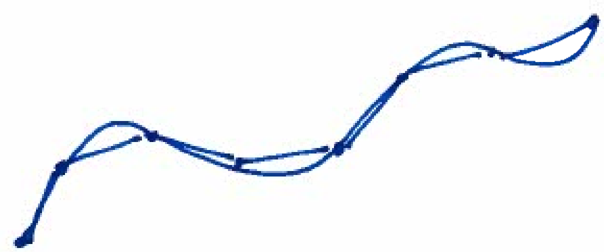
It will sometimes be convenient to allow $d(x_1, x_2) = \infty$: this corresponds to x_1, x_2 being in different connected components of X.

In Riemannian geometry, we define the distance between two points of a Riemannian manifold to be the infimum of the lengths of paths that connect the two points. We want to extend these ideas to the metric space category.

Definition: Let $\gamma: [0,1] \to X$ be a path in a metric space X. We define the <u>length</u> of γ by the formula

$$\ell(\gamma) = \sup \left\{ \sum_i d(\gamma(t_i), \gamma(t_{i+1})) \right\},$$

where the supremum is taken over all partitions
$0 = t_0 < t_1 < \cdots < t_N = 1$ of $[0,1]$.



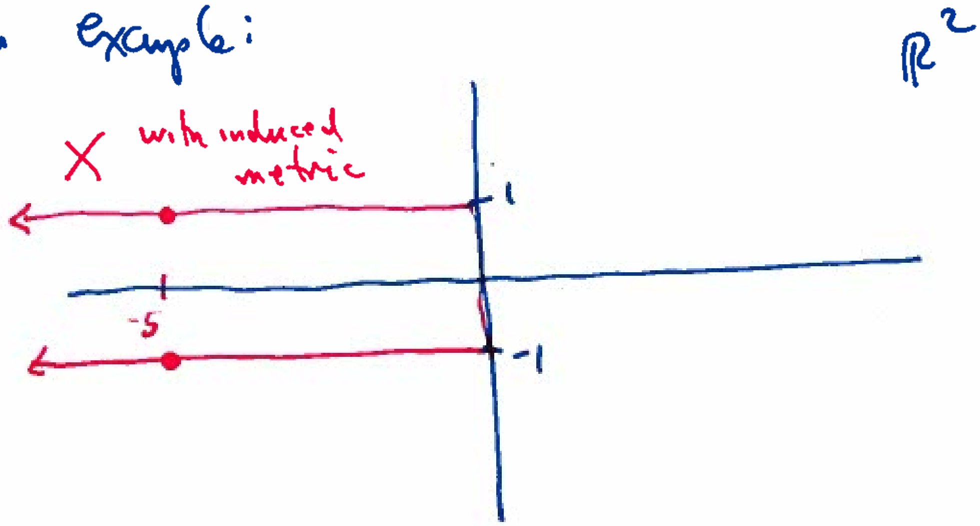
Note that it is possible for $\ell(\gamma) = \infty$.

Definition: A connected metric space X is a <u>length space</u> or a <u>path space</u> if the distance between any two points is the infimum of the lengths of the paths that connect them.

Examples: • $\mathbb{R}^n$

• Any connected Riemannian manifold (by definition, almost)

Not an example:



$d((-5,-1),(-5,1)) = 2$, but the shortest path joining these points has length 12.

Definition: A ~~path~~ curve $\gamma : [0,a] \to X$ is a <u>geodesic segment</u> if γ is an isometry.

Remark 1: Being a geodesic segment is a stronger — and more global — condition than the "locally length minimizing" condition one sees in differential geometry.

Remark 2: If γ is a geodesic segment, then the length of γ equals the distance (in X) between its endpoints.

Remark 3: Conversely, If γ is any curve whose length is equal to the distance between its endpoints, then γ can be reparametrized (if necessary) to become a geodesic segment.

More definitions:

- A <u>geodesic ray</u> in X is an isometry of $[0,\infty)$ into X;
- A <u>geodesic</u> is an isometry of $\mathbb{R}$ into X;
- X is a <u>geodesic space</u> if every pair of points can be joined by a geometric segment.

Remark: Every geodesic space is a length space, but not conversely:

Counterexample: $\mathbb{R}^2 - \{(0,0)\}$.

If (X,d) is not a length space, we can make it into a length space in the following way: for $x, x' \in X$, define $\delta(x,x')$ to be the infimum of the (d-) lengths of paths joining x and x'. Then δ is a metric on X, and makes (X,δ) into a length space: the d-length of any curve equals its δ-length.

There is a natural map $(X,\delta) \to (X,d)$, & this map is continuous, but it is <u>not</u> necessarily a homeomorphism - consider a metric space X containing points x, x' that can only be connected by a non-rectifiable curve.

curve of infinite length

We call δ the <u>induced length metric</u> on X.