Next time (remind definition of bounded + controlled)

Proposition: Let X be a coarse space

(a) If B is a bounded subset of X and $E \subset X \times X$ is controlled, then $E[B]$ is bounded.

(b) If Y, Z are bounded subsets of X with nonempty intersection, then $Y \cup Z$ is bounded.

Given ~~X is a~~ a coarse space X, decree that $x \sim \tilde{x}$ if $\{x, \tilde{x}\}$ is a bounded subset of X.

This is an equivalence relation on X, +

$[x]$ = <u>coarsely connected component</u> of x in X.

We can extend our earlier definitions on metric spaces to coarse spaces:

Definitions: Let X, Y be coarse spaces, $f: X \to Y$ a map.

(a) f is proper if $f^{-1}(B)$ is bounded in X for every bounded subset B of Y.

(b) f is bornologous if $(f \times f)(E)$ is a controlled subset of $Y \times Y$ for every controlled subset E of $X \times X$.

(c) f is coarse if it is proper and bornologous.

(d) X, Y are coarsely equivalent if there exist coarse maps $f: X \to Y$, $g: Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y.

The Coarse Category:

- Objects are the class of all coarse spaces
- Morphisms are coarse maps, with close coarse maps ~~identify~~ identified.

If X is both a topological space and a coarse space, we would like some compatibility between the two structures.

Definition: Let X be a paracompact Hausdorff top space. A coarse structure on X is <u>proper</u> if

(i) there is a controlled nbhd of the diagonal;

(ii) every bounded subset of X has compact closure.

Remark: These conditions imply that X is locally compact.

Proposition: Let X be a connected top space equipped with a proper coarse structure. Then X is coarsely connected. A subset of X is bounded iff it has compact closure, and every controlled subset of $X \times X$ is proper (by our definition at the beginning of this chapter)

Proposition: Let (X,d) be a metric space. Its bounded coarse structure is proper if & only if it is proper as a metric space; i.e., closed bounded sets are compact.

Compactifications - As Viewed by an Operator Algebraist!

- A top space X is <u>locally compact at $x \in X$</u> if there exists a nbhd of x with compact closure. If X is locally compact at each of its points, we say X is <u>locally compact</u>.

- A <u>compactification</u> of a top space X is a compact top space Y such that
  - X is a subspace of Y;
  - $\bar{X} = Y$.

- A (Hausdorff) top space X admits a compactification if & only if X is locally compact.

· Let X be a top space. A function $f: X \to \mathbb{C}$ <u>vanishes at infinity</u> if for every $\varepsilon > 0$ there exists a compact subset K of X such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$.

~~$C_0(X)$~~ Let X be a locally compact Hausdorff space. We let $C_0(X)$ denote the collection of continuous $\mathbb{C}$-valued functions on X that vanish at infinity. $C_0(X)$ is a normed algebra with norm

$$\|f\| := \sup\{|f(x)| : x \in X\},$$

and has a unit if & only if X is actually compact.

Next, let $C_b(X)$ be the collection of continuous functions $f: X \to \mathbb{C}$ with the property that

$$\sup\{|f(x)| : x \in X\} < \infty.$$

This is a unital normed algebra that contains $C_0(X)$ as an ideal. Note that $C_b(X) = C_0(X)$ if & only if X is compact. Also note $C_b(X)$, $C_0(X)$ are closed under complex conjugation.

Now suppose $\mathcal{A}$ is a unital normed-closed subalgebra of $C_b(X)$ that is closed under complex conjugation + contains $C_0(X)$.

By the Gelfand-Naimark Theorem,

$$\mathcal{A} \cong C(Y)$$

where Y is a compact Hausdorff space. Specifically,

Y = space of mult. lin. functionals $Y \to \mathbb{C}$ in the w*-topology (topology of pointwise convergence)

$\boxed{Y \text{ is the maximal ideal space of } \mathcal{A}}$

Observe that X sits naturally inside of Y as evaluation maps: $\phi_x : \mathcal{A} \to \mathbb{C}$,

$$\phi_x(f) = f(x).$$

Moreover, $\overline{X} = Y$, so $\mathcal{A}$ determines a compactification of X. ~~[illegible]~~

Examples:

1. $\mathcal{A} = \{f \in C_b(X) : f - \lambda \in C_0(X) \text{ for some } \lambda \in \mathbb{C}\}$

   $Y = X^+$, the one-point compactification of X

2. $\mathcal{A} = C_b(X)$

   $Y = \beta X$, the Stone-Čech compactification of X

3. $X = \mathbb{R}$, $\mathcal{A} = \{f : \mathbb{R} \to \mathbb{C},\ \lim_{x \to \infty} f(x), \lim_{x \to -\infty} f(x) \text{ exist}\}$

   $Y = [-\infty, \infty]$

Next time:

Theorem: Let X be paracompact and locally compact Hausdorff, & take $E \subset X \times X$. Let $\overline{X}$ be a compactification of X, & let $\partial X := \overline{X} \setminus X$.

TFAE: