Index Theory:

Abstract idea: Suppose A is an algebra, B is an ideal in A, & $a \in A$ is invertible mod B. Then a has an <u>index</u> in $K_0(B)$

$$\cdots \longrightarrow K_1(A) \longrightarrow K_1(A/B) \xrightarrow{\partial} K_0(B)$$

We want to choose A, B to be geometrically interesting, with the following restrictions on B:

- B must be large enough so that interesting elements of A are invertible mod B
- B must be small enough that we can "control" $K_0(B)$.

Example: $\mathcal{H}$ - ~~Hilbert~~ infinite dim'l separable Hilbert space

$\mathcal{L}(\mathcal{H})$ - algebra of bounded operators $T: \mathcal{H} \to \mathcal{H}$

$\mathcal{K}(\mathcal{H})$ - ideal of compact operators

$T: \mathcal{H} \to \mathcal{H}$ is <u>Fredholm</u> if T is invertible mod $\mathcal{K}$

$K_0(\mathcal{K}) \cong \mathbb{Z}$;

$$\operatorname{index} T = [P_{\ker T}] - [P_{\ker T^*}].$$

Atiyah-Singer computes this index for elliptic differential operators on a <u>compact</u> manifold; the index is computed in terms of geometric data about the manifold and the principal symbol of the operator.

What about index theory for <u>noncompact</u> (open) manifolds?

Definition: Let M be a locally compact Hausdorff space, and let H be a Hilbert space. We call H an <u>M-module</u> if it is equipped with a representation $\rho: C_0(M) \to \mathcal{B}(\mathcal{H})$. We say that $\mathcal{H}$ is standard if ρ is faithful and $\rho(C_0(M))$ contains no nonzero compact operator.

Example: M a (noncompact) manifold, S a complex Hermitian vector bundle over M, $H = L^2(S)$, the Hilbert space of sections of S.

$$\rho: C_0(M) \to \mathcal{B}(L^2(S))$$
$$\rho(f)s = f \cdot s$$

$L^2(S)$ is a <u>standard</u> M-module.

Definition: Let H be an M-module, & take $v \in H$. The <u>support</u> $\mathrm{Supp}(v)$ of v is the smallest closed subset K of M with the property that if $f \in C_0(M)$ & $f|_K \equiv 0$, then $f \cdot v = 0$.

Proposition: A point $x \in M$ belongs to $\mathrm{Supp}(v)$ if & only if $g \cdot v \neq 0$ for any function g that is 1 in a neighborhood of x.

Definition: Let X be a metric space, & let Y be a closed subspace of X. For each $R > 0$, define

$$\mathrm{Pen}(Y; R) = \{x \in X : d(x, Y) \leq R\}.$$

Definition: Let $\mathcal{H}$ be an M-module & let $A \in \mathcal{B}(\mathcal{H})$. We say A has <u>bounded propagation</u> if there exists $R > 0$ such that

$$\mathrm{Supp}(Av) \cup \mathrm{Supp}(A^* v) \subseteq \mathrm{Pen}(\mathrm{Supp}(v); R).$$

for all $v \in \mathcal{H}$.

Let $A_{\mathcal{H}}$ be the collection of all bounded propagation operators on $\mathcal{H}$. $A_{\mathcal{H}}$ is a unital $*$-algebra & $\rho(C_0(M)) \subseteq A_{\mathcal{H}}$.

Definition: Let H be an M-module. A positive operator $A \in \mathcal{B}(\mathcal{H})$ is locally traceable if $\rho(f) A \rho(f)$ is trace class for every compactly supported function f on M. A general (i.e., not nec. positive) operator is locally traceable if it is a finite linear combination of positive locally traceable operators.

Proposition: The collection $\mathcal{B}_H$ of all locally traceable operators with bounded propagation is a $*$-ideal in A_H.

Definition: Let H be an M-module, & suppose D is an unbounded self-adjoint operator on $\mathcal{H}$. We say D is a (generalized) elliptic operator on $\mathcal{H}$ if

- $\exists\ c > 0$ such that $e^{itD} \in A_H$ $\forall t \in \mathbb{R}$ & has propagation bound $\leq c|t|$.
- $(1+D^2)^{-n}$ is locally traceable for some $n > 0$.

Definition: A smooth function $\psi : \mathbb{R} \to \mathbb{R}$ is a chopping function if

$$\lim_{x \to \pm\infty} \psi(x) = \pm 1$$

and ψ' has compactly supported fourier transform.

Proposition: Let D be an elliptic operator on a $\mathcal{M}$-module H. Then $\psi(D) \in A_H$ for any chopping function ψ, & $\psi(D)$ is invertible modulo B_H.

This allows us to define $\operatorname{ind}(D) \in K_0(B_H)$.

Proposition: Suppose 0 is not in the spectrum of D. Then $\operatorname{ind}(D) = 0$.

General problem: we don't know very much about $K_0(B_H)$; in particular, how to compute it.

Idea: Suppose we have a coarse cocycle $\phi \in ZX^{2q}(M)$, & hence $[\phi] \in HX^{2q}(M)$. A. Connes has defined a character map

$$\chi : HX^{2q}(M) \to HC^{2q}(B_H),$$

where HC^* represents cyclic cohomology. Cyclic cohomology naturally pairs with K-theory, so we have a number

$$\mathrm{Ind}_\phi(D) = \langle \mathrm{Ind}(D), \chi[\phi] \rangle$$

Index Problem: Compute $\mathrm{Ind}_\phi(D)$ in terms of geometric information about ϕ and D.

Proposition: If $c[\phi] \in H_c^{2q}(M)$ is zero, then $\mathrm{Ind}_\phi(D) = 0$.

Theorem: Let D be a Dirac operator over an even-dim'l manifold M. Then

$$\mathrm{Ind}_\phi(D) = \frac{q!}{(2q)!(2\pi)^2}\left(\hat{A}(M) \cup ch(E) \cup c[\phi], [M] \right).$$

Consequence of Roe's Index Theorem:

Definition: Let M be a complete oriented n dim'l Riemannian manifold. We say M is ultraspherical if there is an essential $(n-1)$ sphere in its Higson Corona; in other words, there is a map $\sigma : \partial M \to S^{n-1}$ + $\theta \in H^{n-1}(S^{n-1})$ denote the fundamental class, then $\langle \partial \sigma^* [\theta], [M] \rangle \neq 0$.

Examples:

- $\mathbb{R}^n$
- Any simply connected complete manifold that admits a metric of non-positive curvature.

Proposition: M is ultraspherical $\Leftrightarrow$ there exists a proper smooth map from M to $\mathbb{R}^n$ that has nonzero degree + a uniformly bounded gradient.

Definition: A compact connected manifold is U-enlargeable if it has a covering that is spin and ultraspherical.

Example: If M admits a metric ~~of~~ of non-positive sectional curvature, then M is Λ-enlargeable.

Theorem: Let M be a compact manifold, and suppose there is a spin map of nonzero $\hat{A}$-degree from M to ~~on~~ a Λ-enlargeable manifold M_0. Then M does not admit a metric of positive scalar curvature.

Corollary (Gromov–Lawson): A Λ-enlargeable manifold ~~does not~~ does not admit a metric of positive ~~scal~~ scalar curvature.