Traces, Determinants, and Toeplitz Operators

Let A be an $n \times n$ matrix with complex entries:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Then

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$$

and

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Properties of trace: For A and B in $\mathrm{M}(n, \mathbb{C})$ and S in $\mathrm{GL}(n, \mathbb{C})$,

- $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$;
- $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
- $\operatorname{tr}(SAS^{-1}) = \operatorname{tr} A$;
- The trace of A is the sum of the eigenvalues of A.

Properties of determinant:

- $\det(AB) = \det(BA) = (\det A)(\det B)$;
- $\det(SAS^{-1}) = \det A$;
- The determinant of A is the product of the eigenvalues of A.

Define the *exponential* of A as

$$\exp A := \sum_{k=0}^{\infty} \frac{1}{k\,!} A^k.$$

Warning: In general $\exp(A + B) \neq (\exp A)(\exp B)$ unless A and B commute.

Theorem: $\det(\exp A) = e^{\operatorname{tr} A}$

Let V be a complex vector space equipped with an inner product. This is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all elements v, w, and u in V and all complex numbers α and β,

- $\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, u \rangle$;
- $\langle v, \alpha w + \beta u \rangle = \overline{\alpha} \langle v, w \rangle + \overline{\beta} \langle v, u \rangle$;
- $\langle w, v \rangle = \overline{\langle v, w \rangle}$;
- $\langle v, v \rangle \geq 0$, with $\langle v, v \rangle = 0$ if and only if $v = 0$.

An *orthonormal basis* for V is a vector space basis $\{e_k\}_{k=1}^n$ for V with the additional properties

- $\langle e_k, e_k \rangle = 1$ for $1 \leq k \leq n$;
- $\langle e_k, e_\ell \rangle = 0$ for $k \neq \ell$.

Let A be a linear transformation of V. Then

$$\operatorname{tr} A = \sum_{k=1}^{n} \langle Ae_k, e_k \rangle$$

and

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) \left\langle Ae_1, e_{\sigma(1)} \right\rangle \left\langle Ae_2, e_{\sigma(2)} \right\rangle \cdots \left\langle Ae_n, e_{\sigma(n)} \right\rangle .$$

These quantities are independent of the choice of orthonormal basis.

The *adjoint* of A is the linear transformation determined by the equation

$$\langle Av, w \rangle = \langle v, A^* w \rangle$$

for all v and w in V.

If we write A as a matrix with respect to an orthonormal basis, then A^* is the complex conjugate transpose of A; i.e., the (i, j) entry of A^* is $\overline{a_{ji}}$. Thus

$$\operatorname{tr} A^* = \overline{\operatorname{tr} A}, \qquad \det A^* = \overline{\det A}.$$

Now let V be an infinite-dimensional complex inner product space and define a norm $\|v\| := \sqrt{\langle v, v\rangle}$ for every v in V. We say that V is *complete* if every Cauchy sequence with respect to this norm is convergent. In this case we will use the letter $\mathcal{H}$ to denote our complex inner product space, and we call it a *Hilbert space*.

We will only consider *separable* Hilbert spaces. This means that $\mathcal{H}$ contains a countably infinite subset $\{e_k\}$ with the following properties:

- $\langle e_k, e_k\rangle = 1$ for all k;
- $\langle e_k, e_\ell\rangle = 0$ for $k \neq \ell$;
- $v = \sum_{k=1}^{\infty} \langle v, e_k\rangle e_k$ for every v in $\mathcal{H}$.

Warning: the set $\{e_k\}$ is **not** a vector space basis!

Let A be a linear transformation of $\mathcal{H}$. We say that A is *bounded* if

$$\|A\| := \sup\left\{\frac{\|Av\|}{\|v\|} : v \neq 0\right\} < \infty.$$

We will call a bounded linear transformation of $\mathcal{H}$ an *operator* on $\mathcal{H}$.

The collection of all operators on $\mathcal{H}$ is an *algebra* (closed under addition, multiplication [composition], scalar multiplication), and is denoted $\mathcal{B}(\mathcal{H})$.

How do we define trace for operators on $\mathcal{H}$?

Naive idea: choose an orthonormal basis $\{e_k\}$ for $\mathcal{H}$ and set

$$\operatorname{tr} A = \sum_{k=1}^{\infty} \langle Ae_k, e_k\rangle\,.$$

Problem 1: The right-hand side does not necessarily converge.

Example:

$$\operatorname{tr} I = \sum_{k=1}^{\infty} \langle Ie_k, e_k\rangle = \sum_{k=1}^{\infty} \langle e_k, e_k\rangle = \sum_{k=1}^{\infty} 1 = \infty.$$

So not every operator has a well-defined trace.

Problem 2: Even if the right-hand side does converge, its value may depend on the choice of orthonormal basis.

An operator P on $\mathcal{H}$ is *positive* if $\langle Pv, v\rangle \geq 0$ for all v in $\mathcal{H}$.

Example: Let A be any operator on $\mathcal{H}$. Then A^*A is positive, because

$$\langle A^*Av, v\rangle = \langle Av, Av\rangle \geq 0.$$

In fact, every positive operator P has this form for some operator A.

If P is positive, then $\sum_{k=1}^{\infty} \langle Pe_k, e_k\rangle$ is in $[0, \infty]$ and is independent of the choice of orthonormal basis.

Every positive operator P has a positive *square root operator* $\sqrt{P}$. Define

$$|A| := \sqrt{A^*A}.$$

Example: Take

$$A = \begin{pmatrix} -\frac{27}{25} + \frac{32}{25}i & -\frac{36}{25} - \frac{24}{25}i \\ -\frac{36}{25} - \frac{24}{25}i & -\frac{48}{25} + \frac{18}{25}i \end{pmatrix}.$$

Then

$$A^*A = \begin{pmatrix} \frac{29}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{36}{5} \end{pmatrix}.$$

Let

$$S = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

Then

$$S^{-1}(A^*A)S = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix},$$

whence

$$\sqrt{S^{-1}(A^*A)S} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

and thus

$$|A| = S\left(\sqrt{S^{-1}(A^*A)S}\right)S^{-1} = \begin{pmatrix} \frac{59}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{66}{25} \end{pmatrix}.$$

Define

$$\mathcal{L}^1(\mathcal{H}) := \left\{ A \in \mathcal{B}(\mathcal{H}) : \sum_{k=1}^{\infty} \langle |A| e_k, e_k \rangle < \infty \right\}.$$

The set $\mathcal{L}^1(\mathcal{H})$ is an ideal in $\mathcal{B}(\mathcal{H})$ and is called the *ideal of trace-class operators* on $\mathcal{H}$. For A in $\mathcal{L}^1(\mathcal{H})$ we can define $\operatorname{tr} A$ in the naive way we originally proposed:

$$\operatorname{tr} A = \sum_{k=1}^{\infty} \langle A e_k, e_k \rangle .$$

Properties of tr:

- $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$ for A and B in $\mathcal{L}^1(\mathcal{H})$;
- $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for A in $\mathcal{L}^1(\mathcal{H})$ and B in $\mathcal{B}(\mathcal{H})$;
- $\operatorname{tr}(SAS^{-1}) = \operatorname{tr} A$ for A in $\mathcal{L}^1(\mathcal{H})$ and S in $\mathcal{B}(\mathcal{H})$ invertible;
- $\operatorname{tr} A$ is the sum of the eigenvalues of A for all A in $\mathcal{L}^1(\mathcal{H})$.

Remark: This last statement, known as Lidskii's theorem, was not proved until 1959.

How do we define the determinant?

For $\|A\| < 1$, we can define the logarithm of $I + A$ by the infinite series

$$\log(I + A) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^n.$$

If A is trace class, then for $\mu \in \mathbb{C}$ with sufficiently small modulus, the operator $\log(1 + \mu A)$ is also trace class, so we can define

$$\det(I + \mu A) = e^{\operatorname{tr}(\log(I+\mu A))}$$

and then extend by analytic continuation, so that the domain of det is

$$\mathrm{GL}\big(1, (I + \mathcal{L}^1(\mathcal{H}))\big),$$

the multiplicative group of invertible elements of $\mathcal{B}(\mathcal{H})$ of the form $I + L$ for some L in $\mathcal{L}^1(\mathcal{H})$.

Properties of det:

- $\det(AB) = (\det A)(\det B)$ for A and B in $\mathrm{GL}(1, I + \mathcal{L}^1(\mathcal{H}))$;
- $\det A^{-1} = (\det A)^{-1}$ for A in $\mathrm{GL}(1, (I + \mathcal{L}^1(\mathcal{H}))$;
- $\det(SAS^{-1}) = \det A$ for A in $\mathrm{GL}(1, (I + \mathcal{L}^1(\mathcal{H}))$ and S in $\mathcal{B}(\mathcal{H})$ invertible;
- $\det A$ is the product of the eigenvalues of A for A in $\mathrm{GL}(1, I + \mathcal{L}^1(\mathcal{H}))$.

These quantities are hard to compute directly, especially the determinant! However, in certain cases of geometric and/or topological interest, there are other ways to proceed.

Example 1:

Suppose $K : [a, b] \times [a, b] \to \mathbb{C}$ is continuous and define A in $\mathcal{B}(L^2[a, b])$ by the formula

$$(Af)(x) = \int_a^b K(x, y) f(y)\, dy.$$

This is an example of a *compact* operator. It is not always trace class (in fact, it is an open problem to find necessary and sufficient conditions on K so that A is trace class), but if A, is trace class, then

$$\mathrm{tr}\, A = \int_a^b K(x, x)\, dx.$$

We can also express $\det(I + A)$ in terms of K. For each n-tuple $(x_1, x_2, \ldots, x_n)$ in $[a, b]$, define

$$K_n(x_1, x_2, \ldots, x_n) = \det \begin{pmatrix} K(x_1, x_1) & K(x_1, x_2) & \cdots & K(x_1, x_n) \\ K(x_2, x_1) & K(x_2, x_2) & \cdots & K(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, x_1) & K(x_n, x_2) & \cdots & K(x_n, x_n) \end{pmatrix}$$

Then

$$\det(I + A) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_a^b \cdots \int_a^b K_n(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2\, \ldots dx_n.$$

Example 2:
Consider the Hilbert space $L^2(S^1)$ with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta.$$

This Hilbert space has orthonormal basis

$$\{e^{in\theta} : n \in \mathbb{Z}\} = \{z^n : n \in \mathbb{Z}\}.$$

Let $C(S^1)$ denote the algebra of continuous complex-valued functions on the circle. For each ϕ in $C(S^1)$, define an operator M_ϕ on $L^2(S^1)$ via pointwise multiplication:

$$(M_\phi f)(x) = \phi(x) f(x).$$

Next, let $H^2(S^1)$ be the Hilbert subspace of $L^2(S^1)$ whose orthonormal basis is

$$\{z^n : n \geq 0\}.$$

An alternate description of $H^2(S^1)$ is the Hilbert subspace of the elements of $L^2(S^1)$ that extend to analytic functions on the disk $\{z \in \mathbb{C} : |z| < 1\}$.

Define the *orthogonal projection* $P : L^2(S^1) \to H^2(S^1)$ by

$$P\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n.$$

Then for each ϕ in $C(T)$, define the *Toeplitz operator* T_ϕ on $H^2(S^1)$ by the formula

$$T_\phi = P M_\phi.$$

Properties of Toeplitz operators: For ϕ and ψ in $C(S^1)$ and λ in $\mathbb{C}$,

- $T_{\phi+\psi} = T_\phi + T_\psi$;
- $T_{\lambda\phi} = \lambda T_\phi$;
- $T_\phi^* = T_{\overline{\phi}}$.

$T_{\phi\psi} \neq T_\phi T_\psi$ in general, but for ϕ and ψ in $C^\infty(S^1)$, we have

$$T_\phi T_\psi - T_\psi T_\phi \in \mathcal{L}^1(\mathcal{H}).$$

Surprisingly (at first), the trace of this quantity can be nonzero. This is because $T_\phi T_\psi$ and $T_\psi T_\phi$ are typically not trace class operators, but their difference is.

Example:

$$T_{z^{-3}}T_{z^3}(z^n) = z^n \text{ for all } n \geq 0$$

$$T_{z^3}T_{z^{-3}}(z^n) = \begin{cases} 0 & 0 \leq n < 3 \\ z^n & n \geq 3 \end{cases}$$

Therefore

$$\text{tr}\,(T_{z^{-3}}T_{z^3} - T_{z^3}T_{z^{-3}}) = 3.$$

In general,

$$\text{tr}\,(T_{z^m}T_{z^n} - T_{z^n}T_{z^m}) = \begin{cases} n & \text{if } m+n=0 \\ 0 & \text{otherwise.} \end{cases}$$

Also observe that

$$\frac{1}{2\pi i}\int_0^{2\pi} e^{im\theta} d(e^{in\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} ine^{im\theta}e^{in\theta}\, d\theta = \begin{cases} n & \text{if } m+n=0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem: For ϕ and ψ in $C^\infty(S^1)$,

$$\text{tr}\,(T_\phi T_\psi - T_\psi T_\phi) = \frac{1}{2\pi i}\int_{S^1} \phi\, d\psi.$$

Proof: Write ϕ and ψ in terms of the basis $\{z^n : n \geq 0\}$ and combine the linearity of the trace and the integral with the computations in the example above. □

We can generalize this result somewhat. Define

$$\mathcal{T}^\infty := \left\{T_\phi + L : \phi \in C^\infty(S^1), L \in \mathcal{L}^1(H^2(S^1))\right\}.$$

Then there exists a short exact sequence

$$0 \longrightarrow \mathcal{L}^1(H^2(S^1)) \longrightarrow \mathcal{T}^\infty \xrightarrow{\ \sigma\ } C^\infty(S^1) \longrightarrow 0\ ,$$

and the *symbol map* $\sigma : \mathcal{T}^\infty \to C^\infty(S^1)$ is given by the formula $\sigma(T_\phi + L) = \phi$.

Theorem: For T and W in $\mathcal{T}^\infty$,

$$\text{tr}\,(TW - WT) = \frac{1}{2\pi i}\int_0^{2\pi} \sigma(T)(\theta)\sigma(W)'(\theta)\, d\theta$$

Now let's look at the determinant.

Take invertible elements T and W in $\mathcal{T}^\infty$, and set $\phi = \sigma(T)$ and $\psi = \sigma(W)$. Then

$$\sigma\left(TWT^{-1}W^{-1}\right) = \phi\psi\phi^{-1}\psi^{-1} = 1,$$

whence $T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}$ is in $I + \mathcal{L}^1(H^2(S^1))$.

$$\det\bigl(TWT^{-1}W^{-1}\bigr) = ??$$

Here is an answer in a very special case. If A is an element of $\mathcal{T}^\infty$, then $\exp A$ is an invertible element of $\mathcal{T}^\infty$ with inverse $\exp(-A)$.

Theorem: For A and B in $\mathcal{T}^\infty$,

$$\det\bigl(\exp A \exp B \exp(-A)\exp(-B)\bigr) = \exp\left(\frac{1}{2\pi i}\int_0^{2\pi} \sigma(A)(\theta)\sigma(B)'(\theta)\,d\theta\right).$$

Let's look at this from a different point of view.

Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{H}^n$ is also a Hilbert space:

$$\langle (v_1, v_2, \ldots, v_n), (w_1, w_2, \ldots, w_n)\rangle := \langle v_1, w_1\rangle + \langle v_2, w_2\rangle + \cdots + \langle v_n, w_n\rangle.$$

We can view elements of $\mathcal{B}(\mathcal{H}^n)$ as elements of $\mathrm{M}(n, \mathcal{B}(\mathcal{H}))$. By extending the notion of symbol in the obvious way, we have a short exact sequence

$$0 \longrightarrow \mathcal{L}^1((H^2(S^1))^n) \longrightarrow \mathrm{M}(n, \mathcal{T}^\infty) \xrightarrow{\ \sigma\ } \mathrm{M}(n, C^\infty(S^1)) \longrightarrow 0\,.$$

Suppose ϕ and ψ are arbitrary invertible elements of $C^\infty(S^1)$. Then we can find matrices R and S in $\mathrm{GL}(3, \mathcal{T}^\infty)$ such that

$$\sigma(R) = \begin{pmatrix} \phi & 0 & 0 \\ 0 & \phi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\sigma(S) = \begin{pmatrix} \psi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \psi^{-1} \end{pmatrix}.$$

For example, we can choose

$$R = \begin{pmatrix} 2T_\phi - T_\phi T_{\phi^{-1}} T_\phi & T_\phi T_{\phi^{-1}} - I & 0 \\ I - T_{\phi^{-1}} T_\phi & T_{\phi^{-1}} & 0 \\ 0 & 0 & I \end{pmatrix}$$

and

$$S = \begin{pmatrix} 2T_\psi - T_\psi T_{\psi^{-1}} T_\psi & 0 & T_\psi T_{\psi^{-1}} - I \\ 0 & I & 0 \\ I - T_{\psi^{-1}} T_\psi & 0 & T_{\psi^{-1}} \end{pmatrix}$$

We infer from the short exact sequence above that the operator $RSR^{-1}S^{-1}$ is determinant-class. Furthermore, the value of this determinant does not depend on the choice of R and S satisfying the properties above - the determinant of $RSR^{-1}S^{-1}$ only depends on ϕ and ψ.

Suppose that ϕ and ψ are restrictions of meromorphic functions (which we also denote ϕ and ψ) defined in a neighborhood of the closed unit disk such that neither ϕ nor ψ has zeros or poles on the unit circle. For each point z in the open unit disk $\mathbb{D}$, define

$$v(\phi, z) = \begin{cases} m & \text{if } \phi \text{ has a zero of order } m \text{ at } z \\ -m & \text{if } \phi \text{ has a pole of order } m \text{ at } z \\ 0 & \text{if } \phi \text{ has neither a zero nor a pole at } z, \end{cases}$$

and similarly define $v(\psi, z)$. The quantity

$$\lim_{w \to z} (-1)^{v(\phi,z)v(\psi,z)} \frac{\psi(w)^{v(\phi,z)}}{\phi(w)^{v(\psi,z)}}$$

is called the *tame symbol* of ϕ and ψ at z and is denoted $(\phi, \psi)_z$.

Example:

$\phi(z) = \dfrac{z^3 - 3z^2}{2z+1}$ double zero at 0, simple zero at 3, simple pole at $-1/2$

$\psi(z) = \dfrac{2z-1}{z^3}$ simple zero at 1/2, triple pole at 0

$$
\begin{aligned}
(\phi,\psi)_0 &= \lim_{w\to 0}\left((-1)^{(2)(-3)}\frac{\left(\frac{2w-1}{w^3}\right)^2}{\left(\frac{w^2(w-3)}{2w+1}\right)^{-3}}\right)\\
&= \lim_{w\to 0}\frac{(2w-1)^2}{w^6}\cdot\frac{w^6(w-3)^3}{(2w+1)^3}\\
&= \lim_{w\to 0}\frac{(2w-1)^2(w-3)^3}{(2w+1)^3}\\
&= -27
\end{aligned}
$$

$$
\begin{aligned}
(\phi,\psi)_{-1/2} &= \lim_{w\to -1/2}\left((-1)^{(-1)(0)}\frac{\left(\frac{2w-1}{w^3}\right)^{-1}}{\left(\frac{w^2(w-3)}{2w+1}\right)^{0}}\right)\\
&= \lim_{w\to -1/2}\frac{w^3}{2w-1}\\
&= \frac{1}{16}
\end{aligned}
$$

$$
\begin{aligned}
(\phi,\psi)_{1/2} &= \lim_{w\to 1/2}\left((-1)^{(0)(-1)}\frac{\left(\frac{2w-1}{w^3}\right)^{0}}{\left(\frac{w^2(w-3)}{2w+1}\right)^{1}}\right)\\
&= \lim_{w\to 1/2}\frac{2w+1}{w^2(w-3)}\\
&= -\frac{16}{5}
\end{aligned}
$$

We will not compute $(\phi,\psi)_3$ for reasons that will be become clear in a minute. For all other complex numbers z, we see that $(\phi,\psi)_z = 1$.

Theorem:

$$\det(RSR^{-1}S^{-1}) = \prod_{z\in\mathbb{D}}(\phi,\psi)_z^{-1}.$$

Remark 1: Suppose that T_Φ and T_ψ are invertible. Then we can take

$$R = \begin{pmatrix} T_\phi & 0 & 0\\ 0 & T_\phi^{-1} & 0\\ 0 & 0 & I\end{pmatrix}$$

and

$$S = \begin{pmatrix} T_\psi & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T_\psi^{-1} \end{pmatrix},$$

whence

$$\det(RSR^{-1}S^{-1}) = \det \begin{pmatrix} T_\phi T_\psi T_\phi^{-1} T_\psi^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \det \left(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1} \right).$$

Remark 2: In fact, $\det(RSR^{-1}S^{-1})$ only depends on the *Steinberg symbol* $\{\phi, \psi\}$ of ϕ and ψ. This is an element of the algebraic K-theory group $K_2(C^\infty(S^1))$, and we can use the above theorem to prove that certain Steinberg symbols are nontrivial.

Surprising fact that comes out of this circle of ideas: if both ϕ and $\psi := 1 - \phi$ are invertible, then $\det(RSR^{-1}S^{-1}) = 1$.

von Neumann Algebras

Definition: Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra on $\mathcal{H}$ is subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ such that

- A is a $*$-algebra; that is, if A is in $\mathcal{A}$, then A^* is in $\mathcal{A}$
- $\mathcal{A}$ is closed under the topology of pointwise convergence

Example 1: Let X be a locally compact Hausdorff space and let μ is a Borel measure on X. Then $L^\infty(X, \mu)$ is an abelian von Neumann algebra. Furthermore, every abelian von Neumann algebra arises in this manner.

Example 2: Let G be a discrete group. For each g in G, define $\lambda_g : \ell^2(G) \to \ell^2(G)$ by the formula $\lambda(x) = gx$ for every x in G. Then $L^\infty(G)$ is the von Neumann algebra on $\ell^2(G)$ generated by the set $\{\lambda_g : g \in G\}$.

Open Question: Let F_n denote the free group on n generators. Is $L^\infty(F_2)$ isomorphic to $L^\infty(F_3)$?

A von Neumann algebra $\mathcal{A}$ is a *factor* if its center is $\mathbb{C}$.

Theorem: Every factor admits a trace on projections, and the range of this trace is exactly one of the following sets.

- $\{1, 2, 3, \ldots, n\}$, $\quad 1 \leq n \leq \infty$ type I_n factor
- $[0, 1]$ type II_1 factor
- $[0, \infty)$ type II_∞ factor
- $\{0, 1\}$ type III factor

Examples:

- $M(n, \mathbb{C})$ is a type I_n factor, $1 \leq n < \infty$.
- If $\mathcal{H}$ is separable, then $\mathcal{B}(\mathcal{H})$ is a I_∞ factor.
- If G is a group with the property that every nontrivial conjugacy class is infinite, then $L^\infty(G)$ is a II_1 factor.

$$\mathrm{tr}\Big(\sum_{g \in G} a_g \lambda_g\Big) = a_e$$

- $L^\infty(\mathbb{R})$ is a II_∞ factor.

$$\mathrm{tr}(f) = \int_{-\infty}^{\infty} f(x)\, dx$$

- Type III factors: you don't want to know.

Fun With K-theory!

Definition: Let A be a unital Banach algebra with norm $\|\cdot\|_A$ and let J be a not necessarily closed ideal in A. We say that (A, J) is a *relative pair of Banach algebras* if there exists a norm $\|\cdot\|_J$ on J such that

1. the ideal J is a Banach algebra in the norm $\|\cdot\|_J$;
2. for all j in J,
$$\|j\|_J \leq \|j\|_A;$$
3. for all a and b in A and j in J,
$$\|ajb\|_J \leq \|a\|_A \|j\|_J \|b\|_A.$$

A morphism between relative Banach pairs (A, J) and $(\widetilde{A}, \widetilde{J})$ is a continuous algebra map $\omega : (A, \|\cdot\|_A) \to (\widetilde{A}, \|\cdot\|_{\widetilde{A}})$ that restricts to a continuous map $\omega_{|J}$: $(J, \|\cdot\|_J) \to (\widetilde{J}, \|\cdot\|_{\widetilde{J}})$.

Prototypical example: a type II_∞ factor.

If (A, J) is a relative pair of Banach algebras, then $(\mathrm{M}(n, A), \mathrm{M}(n, J))$ is also a relative pair of Banach algebras if we define

$$\|a\|_{\mathrm{M}(n,A)} = \sum_{k,\ell=1}^{n} \|a_{k\ell}\|_A,$$

and similarly define $\|j\|_{\mathrm{M}(n,J)}$ for j in $\mathrm{M}(n, J)$.

For each natural number n, define

$$\mathrm{GL}(n, J) = \{G \in \mathrm{GL}(n, J^+) : G - I_n \in \mathrm{M}(n, J)\}.$$

Let $\mathrm{GL}(n, J)_0$ denote the connected component of the identity, and define

$$K_1^{\mathrm{top}}(J) = \lim_{n\to\infty} \frac{\mathrm{GL}(n, J)}{\mathrm{GL}(n, J)_0}.$$

Next, define

$$[\mathrm{GL}(n,J),\mathrm{GL}(n,A)] = \left\{GHG^{-1}H^{-1} : G \in \mathrm{GL}(n,J), H \in \mathrm{GL}(n,A)\right\}.$$

Then $[\mathrm{GL}(n,J),\mathrm{GL}(n,A)]$ is a normal subgroup of $\mathrm{GL}(n,J)$ for each natural number n. Define

$$K_1^{\mathrm{alg}}(A,J) = \lim_{n\to\infty} \frac{\mathrm{GL}(n,J)}{[\mathrm{GL}(n,J),\mathrm{GL}(n,A)]}.$$

Let $\mathrm{R}(n,J)$ denote the set of smooth paths $\gamma : [0,1] \to \mathrm{GL}(n,J)$ with the property that $\gamma(0) = 1$, and similarly define $\mathrm{R}(n,A)$. These sets are groups under pointwise multiplication, and

$$[\mathrm{R}(n,J),\mathrm{R}(n,A)] = \{\gamma\beta\gamma^{-1}\beta^{-1} : \gamma \in \mathrm{R}(n,J), \beta \in \mathrm{R}(n,A)\}$$

is a normal subgroup of $\mathrm{R}(n,J)$.

Define an equivalence relation $\sim$ on $\mathrm{R}(n,J)$ by decreeing that $\gamma_0 \sim \gamma_1$ if there exists a smooth homotopy $\{\gamma_t\}$ from γ_0 to γ_1 such that $\gamma_t(0) = \gamma_0(0)$ and $\gamma_t(1) = \gamma_0(1)$ for all $0 \le t \le 1$. Let q denote the quotient map from $\mathrm{R}(n,J)$ to the set of equivalence classes of $\sim$, and set

$$K_1^{\mathrm{rel}}(A,J) = \lim_{n\to\infty} \frac{q(\mathrm{R}(n,J))}{q([\mathrm{R}(n,J),\mathrm{R}(n,A)])}.$$

These four groups fit into an exact sequence

$$K_0^{\mathrm{top}}(J) \xrightarrow{\ \partial\ } K_1^{\mathrm{rel}}(A,J) \xrightarrow{\ \theta\ } K_1^{\mathrm{alg}}(A,J) \xrightarrow{\ p\ } K_1^{\mathrm{top}}(J) \longrightarrow 0,$$

with $\theta[\gamma] = [\gamma(1)^{-1}]$ and $p[g] = [g]$.

Suppose that J admits a continuous linear functional $\tau : J \to \mathbb{C}$ with the property that

$$\tau(ja) = \tau(aj)$$

for all j in J and a in A; this is called a *hypertrace*.

Associated to τ is a group homomorphism $\widetilde{\tau}$ from $K_1^{\mathrm{rel}}(A, J)$ to $\mathbb{C}$ that is defined in the following way: let γ be an element of $\mathrm{R}(1, J)$ and let $[\gamma]$ be the corresponding element of $K_1^{\mathrm{rel}}(A, J)$. Then

$$\widetilde{\tau}[\gamma] = -\tau\left(\int_0^1 \gamma'(t)\gamma(t)^{-1}\, dt\right) = -\int_0^1 \tau\left(\gamma'(t)\gamma(t)^{-1}\right)\, dt.$$

Definition: Let $\underline{\tau} : K_0^{\mathrm{top}}(J) \to \mathbb{C}$ be the group homomorphism induced by τ. The *relative de la Harpe-Skandalis determinant* associated to τ is the group homomorphism

$$\widetilde{\det}_\tau : \mathrm{im}(\theta) = \ker(p) \longrightarrow \frac{\mathbb{C}}{2\pi i \cdot \mathrm{im}(\underline{\tau})}$$

that is defined as follows. Suppose g in $\mathrm{GL}(n, J)$ has the property that its class $[g]$ in $K_1^{\mathrm{alg}}(A, J)$ is in the image of θ. Choose β in $\mathrm{R}(n, J)$ so that $\beta(1) = g^{-1}$. Then

$$\widetilde{\det}_\tau[g] = \widetilde{\tau}[\beta] + 2\pi i \cdot \mathrm{im}(\underline{\tau}).$$

If $\mathcal{N}$ is a type II_∞ factor, then its trace τ is a hypertrace, and the pair $(\mathcal{N}, L^1(\tau))$ is a relative pair of Banach algebras. The group $K_1^{\mathrm{top}}(L^1(\tau))$ is trivial whence $\ker(p) = K_1^{\mathrm{alg}}(\mathcal{N}, L^1(\tau))$. Because the trace of any projection in $\mathrm{M}(n, \mathcal{N})$ is real, we see that $\underline{\tau}(K_0^{\mathrm{top}}(L^1(\tau)))$ is contained in $\mathbb{R}$. Therefore, by expanding the codomain of $\widetilde{\det}_\tau$, we have the group homomorphism

$$\widetilde{\det}_\tau : K_1^{\mathrm{alg}}(\mathcal{N}, L^1(\tau)) \to \mathbb{C}/(2\pi i \cdot \mathbb{R}) = \mathbb{C}/i\mathbb{R}.$$

Observe that the map $z + i\mathbb{R} \mapsto e^{\mathrm{Re}(z)}$ is a group isomorphism from $\mathbb{C}/i\mathbb{R}$ to $(0, \infty)$. Composing this isomorphism with $\widetilde{\det}_\tau$, we arrive at the following definition.

Definition: The *semifinite Fuglede-Kadison determinant* for $(\mathcal{N}, L^1(\tau))$ is the group homomorphism

$$\det_\tau : K_1^{\mathrm{alg}}(\mathcal{N}, L^1(\tau)) \longrightarrow (0, \infty)$$

given by the formula

$$\det_\tau[g] = \exp\left(\mathrm{Re}(\widetilde{\det}_\tau[g])\right).$$

The semifinite Fuglede-Kadison determinant enjoys the following properties:

1. $\det_\tau[I] = 1$;
2. $\det_\tau[g_1 g_2] = \det_\tau[g_1] \cdot \det_\tau[g_2]$ for all g_1, g_2 in $\mathrm{GL}(n, L^1(\tau))$;
3. $\det_\tau[hgh^{-1}] = \det_\tau[g]$ for all g in $\mathrm{GL}(n, L^1(\tau))$ and h in $\mathrm{GL}(n, \mathcal{N})$.

Toeplitz Operators on Minimal Ergodic Flows

Let X be a separable compact Hausdorff space equipped with a minimal flow $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$; given a point x in X and a real number t, we will write $\alpha_t(x)$ as $x + t$.

Suppose X admits a Borel probability measure μ with the following properties:

1. the support of μ is all of X;
2. the maps α_t are measure-preserving for each real number t;
3. α is ergodic with respect to μ; i.e., if $Y \subseteq X$ has the property that $\alpha_t(Y) = Y$ for every real number t, then $\mu(Y) = 0$ or $\mu(Y) = 1$.

Endow $\mathbb{R}$ with Lebesgue measure and consider the Hilbert space $L^2(X \times \mathbb{R})$ associated with the product measure on $X \times \mathbb{R}$. Given ϕ in $C(X)$, define M_ϕ on $L^2(X \times \mathbb{R})$ by pointwise multiplication:

$$(M_\phi h)(x, s) = \phi(x)h(x, s).$$

Define the Hilbert transform H on $L^2(X \times \mathbb{R})$ by

$$(Hf)(x,t) = \mathrm{PV}\left(\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1}{s} f(x+s, t-s)\, ds\right).$$

Set $P = \frac{1}{2}(I + H)$. Then P is a projection; denote the range of P by $H^2(X \times \mathbb{R})$. For each ϕ in $C(X)$, define the Toeplitz operator $T_\phi : H^2(X \times \mathbb{R}) \to H^2(X \times \mathbb{R})$ by the formula

$$T_\phi = PM_\phi.$$

The Toeplitz algebra associated to the flow α on X is the C^*-subalgebra $\mathcal{T}(X,\alpha)$ of $\mathcal{B}(H^2(X\times\mathbb{R}))$ generated by the set $\{T_\phi : \phi \in C(X)\}$.

The semi-commutator ideal of $\mathcal{T}(X,\alpha)$ is the C^*-ideal $\mathcal{SC}(X,\alpha)$ of $\mathcal{T}(X,\alpha)$ generated by the set $\{T_\phi T_\psi - T_{\phi\psi} : \phi,\psi \in C(X)\}$.

There is a short exact sequence

$$0 \longrightarrow \mathcal{SC}(X,\alpha) \longrightarrow \mathcal{T}(X,\alpha) \xrightarrow{\sigma} C(X) \longrightarrow 0$$

with the feature that $\sigma(T_\phi) = \phi$ for every ϕ in $C(X)$.

The short exact sequence has an isometric linear splitting ξ defined by $\xi(\phi) = T_\phi$. As a consequence, every element of $\mathcal{T}(X,\alpha)$ can be uniquely written in the form $T_\phi + S$ for some ϕ in $C(X)$ and S in $\mathcal{SC}(X,\alpha)$, and $\|T_\phi\|_{op} = \|\phi\|_\infty$ for every ϕ in $C(X)$.

Remark: The action α on X is called *strictly ergodic* if there exists a **unique** probabilty measure on X for which the α_t are measure-preserving. The commutator ideal of $\mathcal{T}(X,\alpha)$ is contained in $\mathcal{SC}(X,\alpha)$, and if the action of α on X is strictly ergodic then these two ideals are equal. But this is not known in general.

For each real number t, define a unitary operator U_t on $L^2(X\times\mathbb{R})$ by the formula

$$(U_t h)(x,s) = h(x+t, t-s).$$

Let $L^\infty(X) \rtimes \mathbb{R}$ be the von Neumann subalgebra of $\mathcal{B}(L^2(X\times\mathbb{R}))$ generated by the M_ϕ and U_t.

Because the action α is ergodic with respect to the measure μ and μ has full support, $L^\infty(X) \rtimes \mathbb{R}$ is a type II_∞ factor and therefore admits a semifinite normal trace τ. The algebra $C_c(X\times\mathbb{R})$ is weakly dense in $L^\infty(X)\rtimes\mathbb{R}$; we scale τ so that

$$\tau(f) = \int_X f(x,0)\,d\mu(x)$$

for every f in $C_c(X\times\mathbb{R})$.

Define

$$L^p(\tau) = \{F \in L^\infty(X) \rtimes \mathbb{R} : \tau\left(|F|^p\right) < \infty\}$$

and set

$$\|S\|_p = \left(\tau\left(|S|^p\right)\right)^{1/p}, \quad S \in L^p(\tau).$$

Each $L^p(\tau)$ is an ideal in $L^\infty(X) \rtimes \mathbb{R}$.

Holder's Inequality: If A and B are in $L^2(\tau)$, then AB is in $L^1(\tau)$, and $\|AB\|_1 \le \|A\|_2 \|B\|_2$.

Proposition: For all S in $L^p(\tau)$ and F in $L^\infty(X) \rtimes \mathbb{R}$,

$$\|SF\|_p \le \|S\|_p \|F\|_{op}, \quad \|FS\|_p \le \|F\|_{op} \|S\|_p.$$

We can decompose $L^2(X \times \mathbb{R})$ as $H^2(X \times \mathbb{R}) \oplus H^2(X \times \mathbb{R})^\perp$, and via this decomposition, we can view $\mathcal{B}(H^2(X \times \mathbb{R}))$ as a subalgebra of $\mathcal{B}(L^2(X \times \mathbb{R}))$:

$$S \hookrightarrow \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathcal{N}$ be the II_∞ factor

$$\mathcal{N} = P\left(L^\infty(X) \rtimes \mathbb{R}\right) P.$$

The trace τ on $L^\infty(X) \rtimes \mathbb{R}$ restricts to $\mathcal{N}$.

A function ϕ in $C(X)$ is *differentiable* with respect to α if the limit

$$\phi'(x) = \lim_{t \to 0} \frac{\phi(x+t) - \phi(x)}{t}$$

exists for each x in X.

Let $C^1(X, \alpha)$ be the set of functions on X that are continuously differentiable with respect to α. This is a Banach algebra in the norm

$$\|\phi\|_{C^1} = \|\phi\|_\infty + \|\phi'\|_\infty.$$

Theorem: The semicommutator $T_\phi T_\psi - T_{\phi\psi}$ is in $L^1(\tau)$ for all ϕ and ψ in $C^1(X, \alpha)$, and

$$\|T_\phi T_\psi - T_{\phi\psi}\|_1 \le \|\phi\|_{C^1} \|\psi\|_{C^1}.$$

Define

$$\mathcal{SC}^1(X,\alpha) = \mathcal{SC}(X,\alpha) \cap L^1(\tau)$$

$$\mathcal{T}^1(X,\alpha) = \{T_\phi + S : \phi \in C^1(X), S \in \mathcal{SC}^1(X,\alpha)\}$$

Theorem: The exact sequence

$$0 \longrightarrow \mathcal{SC}(X,\alpha) \longrightarrow \mathcal{T}(X,\alpha) \xrightarrow{\sigma} C(X) \longrightarrow 0$$

restricts to an exact sequence of algebras

$$0 \longrightarrow \mathcal{SC}^1(X,\alpha) \longrightarrow \mathcal{T}^1(X,\alpha) \xrightarrow{\sigma} C^1(X,\alpha) \longrightarrow 0 \ ,$$

and the linear splitting $\xi(\phi) = T_\phi$ restricts as well, implying that every element of $\mathcal{T}^1(X,\alpha)$ can be uniquely written in the form $T_\phi + S$ for some ϕ in $C^1(X,\alpha)$ and S in $\mathcal{SC}^1(X,\alpha)$.

Theorem: For T and W in $\mathcal{T}^1(X,\alpha)$, the additive commutator $TW - WT$ is in $L^1(\tau)$, and

$$\tau(TW - WT) = -\frac{1}{2\pi i}\int_X \sigma(T)'(x)\sigma(W)(x)\, d\mu(x).$$

Proposition: $(\mathcal{T}^1(X,\alpha), \mathcal{SC}^1(X,\alpha))$ is a relative pair of Banach algebras, and the inclusion map $\iota : (\mathcal{T}^1(X,\alpha), \mathcal{SC}^1(X,\alpha)) \to (\mathcal{N}, L^1(\tau))$ is a morphism of relative Banach pairs.

Define a map $d : \mathrm{GL}(n, \mathcal{SC}^1(X,\alpha)) \to (0,\infty)$ in the following manner. Suppose that Q is an element in $\mathrm{GL}(n, \mathcal{SC}^1(X,\alpha))$. Then Q determines an element of $[Q]$ of $K_1^{\mathrm{alg}}(\mathcal{T}^1(X,\alpha), \mathcal{SC}^1(X,\alpha))$ and $[\iota(Q)]$ is in $K_1^{\mathrm{alg}}(\mathcal{N}, L^1(\tau))$. Thus we can set

$$d(Q) = \det{}_\tau(\iota[Q]).$$

The map d has the following properties:

1. $d(I) = 1$;
2. $d(Q_1Q_2) = d(Q_1)d(Q_2)$ for Q_1 and Q_2 in $\mathrm{GL}(n, \mathcal{SC}^1(X,\alpha))$;
3. $d(GQG^{-1}) = d(Q)$ for Q in $\mathrm{GL}(n, \mathcal{SC}^1(X,\alpha))$ and G in $\mathrm{GL}(n, \mathcal{T}^1(X,\alpha))$.

Therefore d can be considered to be a determinant function.

Henceforth we will restrict to the situation when $n = 1$.

Let G and H be elements of $\mathrm{GL}(1, \mathcal{T}^1(X, \alpha))$. Then

$$\sigma(GHG^{-1}H^{-1}) = \sigma(G)\sigma(H)\sigma(G)^{-1}\sigma(H)^{-1} = 1,$$

whence $GHG^{-1}H^{-1}$ is an element of $\mathrm{GL}(1, \mathcal{SC}^1(X, \alpha))$.

Proposition: The value of $d(GHG^{-1}H^{-1})$ depends only on $\sigma(G)$ and $\sigma(H)$.

In the case where $G = e^T$ and $H = e^W$ for T and W in $\mathcal{T}^1(X, \alpha)$, we can write down a formula for $d(GHG^{-1}H^{-1})$ in terms of $\sigma(T)$ and $\sigma(W)$.

Lemma: For all T and W in $\mathcal{T}^1(X, \alpha)$,

$$e^T W e^{-T} = W + [T, W] + \frac{1}{2!}[T, [T, W]] + \frac{1}{3!}[T, [T, [T, W]]] + \cdots$$

Proposition: For all T and W in $\mathcal{T}^1(X, \alpha)$,

$$\widetilde{\det}_\tau \left[\iota(e^T e^W e^{-T} e^{-W})\right] = \tau(TW - WT) + i\mathbb{R}.$$

Proof: Define $\beta \in R(1, L^1(\tau))$ by the formula

$$\beta(t) = e^{tW} e^T e^{-tW} e^{-T}.$$

We compute

$$\begin{aligned}\beta'(t)\beta(t)^{-1} &= \left(We^{tW}e^Te^{-tW}e^{-T} - e^{tW}e^TWe^{-tW}e^{-T}\right)e^Te^{tW}e^{-T}e^{-tW} \\ &= W - e^{tW}e^TWe^{-T}e^{-tW}.\end{aligned}$$

Because τ is similarity invariant and because e^{tW} commutes with W, we see that

$$\begin{aligned}\widetilde{\tau}[\beta] &= -\int_0^1 \tau\left(\beta'(t)\beta(t)^{-1}\right) dt \\ &= -\int_0^1 \tau\left(W - e^{tW}e^TWe^{-T}e^{-tW}\right) dt \\ &= -\int_0^1 \tau\left(e^{tW}We^{-tW} - e^{tW}e^TWe^{-T}e^{-tW}\right) dt \\ &= -\int_0^1 \tau\left(W - e^TWe^{-T}\right) dt \\ &= -\tau\left(W - e^TWe^{-T}\right).\end{aligned}$$

Use the lemma above to expand $W - e^T W e^{-T}$:

$$W - e^T W e^{-T} = -\left([T,W] + \frac{1}{2!}[T,[T,W]] + \frac{1}{3!}[T,[T,[T,W]]] + \cdots \right).$$

The right side of this equation converges in the norm on $\mathcal{T}^1(X,\alpha)$. Each summand is in $\mathcal{SC}^1(X,\alpha)$, the norm on $\mathcal{T}^1(X,\alpha)$ dominates the $L^1(\tau)$ norm, and τ is continuous in the $L^1(\tau)$ norm. Therefore

$$-\tau\left(W - e^T W e^{-T}\right) = \tau\left([T,W]\right) + \frac{1}{2!}\tau\left([T,[T,W]]\right) + \frac{1}{3!}\tau\left([T,[T,[T,W]]]\right) + \cdots .$$

Because τ is a hypertrace, all of the terms on the right side vanish except the first one, and thus

$$\widetilde{\det}_\tau\left[\iota(e^T e^W e^{-T} e^{-W})\right] = \widetilde{\tau}[\beta] + i\mathbb{R} = \tau(TW - WT) + i\mathbb{R}.$$

Theorem: Let T and W be elements of $\mathcal{T}^1(X,\alpha)$. Then

$$d(e^T e^W e^{-T} e^{-W}) = \exp\left(\frac{1}{2\pi} \int_X \operatorname{Im}\big(\sigma(T)'(x)\sigma(W)(x)\big)\, d\mu(x) \right).$$

Proof:

$$\begin{aligned} d(e^T e^W e^{-T} e^{-W}) &= \det_\tau\big(\iota[e^T e^W e^{-T} e^{-W}]\big) \\ &= \exp\big(\operatorname{Re}(\tau(TW - WT))\big) \\ &= \exp\left(\operatorname{Re}\left(-\frac{1}{2\pi i} \int_X \sigma(T)'(x)\sigma(W)(x)\, d\mu(x) \right)\right) \\ &= \exp\left(\frac{1}{2\pi} \int_X \operatorname{Im}\big(\sigma(T)'(x)\sigma(W)(x)\big)\, d\mu(x) \right). \end{aligned}$$

Connection to Algebraic K-Theory

We can use the previous theorem and the long exact sequence in algebraic K-theory to construct a homomorphism from $K_2^{\text{alg}}(C^1(X,\alpha))$ to $\mathbb{R}$.

Let $\partial : K_2^{\text{alg}}(C^1(X,\alpha)) \to K_1^{\text{alg}}(\mathcal{T}^1(X,\alpha), \mathcal{SC}^1(X,\alpha))$ be the connecting map from the long exact sequence in algebraic K-theory associated to the short exact sequence

$$0 \longrightarrow \mathcal{SC}^1(X,\alpha) \longrightarrow \mathcal{T}^1(X,\alpha) \xrightarrow{\sigma} C^1(X,\alpha) \longrightarrow 0 \,.$$

Define the group homomorphism $\Delta : K_2^{\text{alg}}(C^1(X,\alpha)) \to (0,\infty)$ to be the composition

$$K_2^{\text{alg}}(C^1(X,\alpha)) \xrightarrow{\partial} K_1^{\text{alg}}(\mathcal{T}^1(X,\alpha), \mathcal{SC}^1(X,\alpha)) \xrightarrow{\iota_*} K_1^{\text{alg}}(\mathcal{M}, L^1(\tau)) \xrightarrow{\det_\tau} (0,\infty) \,.$$

Proposition: Let ϕ and ψ be in $C^1(X,\alpha)$, and let $\{e^\phi, e^\psi\}$ denote the Steinberg symbol of e^ϕ and e^ψ. Then

$$\partial\{e^\phi, e^\psi\} = \left[e^{T_\phi} e^{T_\psi} e^{-T_\phi} e^{-T_\psi}\right] .$$

Theorem: Let ϕ and ψ be in $C^1(X,\alpha)$. Then

$$\Delta\left(\{e^\phi, e^\psi\}\right) = \exp\left(\frac{1}{2\pi}\int_X \operatorname{Im}\left(\phi'(x)\psi(x)\right)\, d\mu(x)\right) .$$

Proof:

$$\begin{aligned}\Delta\left(\{e^\phi, e^\psi\}\right) &= \det_\tau\left(\iota\left[e^{T_\phi} e^{T_\psi} e^{-T_\phi} e^{-T_\psi}\right]\right) \\ &= \exp\left(\frac{1}{2\pi}\int_X \operatorname{Im}\bigl(\sigma(T)'(x)\sigma(W)(x)\, d\mu(x)\bigr)\right) .\end{aligned}$$