SMOOTHING THEOREMS IN ALGEBRAIC GEOMETRY

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Abstract. The purpose of this series of talks is to state a few of the standard smoothing theorems in algebraic geometry, such as Hironaka's theorem, Bertini theorems, and smoothing of degeneracy loci of maps between vector bundles. In the context of these results, I'll then explain recent joint work with Prabhakar Rao.

1. Introduction

The main objects of study in algebraic geometry are smooth projective varieties $X \subset \mathbb{P}^n_k$ over an algebraically closed field k (feel free to take $k = \mathbb{C}$). In recent decades, there has been a lot of work done in trying to understand families of these varieties. Unlike in differential geometry or topology, in algebraic geometry we can often construct parameter spaces for these varieties, or moduli spaces, which turn out to be varieties (or schemes) themselves. Even if one is only interested in smooth projective varieties, one is forced to deal with singular degenerations that appear in the boundary of these moduli spaces if they are proper.

Example 1.1. Each conic $X \subset \mathbb{P}^2$ is defined by a nonzero homogeneous polynomial $f(x, y, z)$ of degree two, where x, y, z are homogeneous coordinates for $\mathbb{P}^2$. Two such polynomials f and g define the same curve X if and only if $f = \lambda g$ for some nonzero scalar $f \in k^*$, defining an equivalence relation $f \sim g$. If V is the vector space of homogeneous degree two polynomials with basis $x^2, y^2, z^2, xy, xz, yz$, then all degree two curves are parametrized by $(V - \{0\}/\sim) = \mathbb{P}^5$, the map is given by

$$f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz \mapsto (a, b, c, d, e, f)$$

This is a simple example of a Hilbert scheme.

We may express f uniquely in the form $f = (x, y, z)M(x, y, z)^T$, where M is a symmetric matrix. Orthogonally diagonalizing M gives a diagonal matrix, thus there is a change of coordinates for which $f(x, y, z) = x^2 + y^2 + z^2, x^2 + y^2$ or x^2. Each possibility can be visualized geometrically:

(A) Writing $x^2+y^2+z^2 = (x+iy)(x-iy)-(iz)^2 = XY-Z^2$ gives another description of a conic in family (A). Some calculation shows that $Z(XY - Z^2) \subset \mathbb{P}^2$ is exactly the image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ given by $(s, t) \mapsto (s^2, t^2, st)$, hence the conic in family (A) is isomorphic to $\mathbb{P}^1$, a smooth rational curve.

(B) In this case we have the equation $XY = 0$, which gives two lines meeting at a point. This conic is not smooth, being singular at the intersection point, it is also not a variety (see definition below), it is a union of two varieties.

(C) The equation $X^2 = 0$ is a doubling of the Y-axis. Classically this might be discounted or maybe thought of as a line, but using Grothendieck's foundations of scheme theory, it gives a closed subscheme of $\mathbb{P}^2$ supported on a line, but having more "scheme structure", i.e. having a different structure sheaf of locally rational functions.

Thus using our parameter space we can stratify our conics as $\mathbb{P}^5 = (A) \cup (B) \cup (C)$. What is the nature of this stratification? If $(\mathbb{P}^2)^\vee$ is the space of lines in $\mathbb{P}^2$, which is parametrized by $\mathbb{P}^2$ itself via $ax + by + cz = 0 \mapsto (a, b, c)$, we have a map $\Phi : (\mathbb{P}^2)^\vee \times (\mathbb{P}^2)^\vee \to \mathbb{P}^5$ given by

$$(L_1, L_2) \mapsto L_1 \cup L_2$$

This is a 2-1 map away from the diagonal, so the image of Φ has dimension 4 and consists of $(B) \cup (C)$. We can do better than this, however: $(B) \cup (C)$ is precisely the locus of $\mathbb{P}^5$ where $\det M = 0$, which gives an equation of degree three, therefore (B) is a cubic hypersurface. Furthermore, (C) is given by the vanishing of the three 2×2 minors of M, hence is the intersection of three quadric hypersurfaces and since $\dim C = 2$, it is a complete intersection, so $(C) \subset \mathbb{P}^5$ is a complete intersection of three quadrics and $\deg(C) = 8$. I suspect that (B) is a nonsingular hypersurface except along (C). It follows that (A) is a dense Zariski open subset of $\mathbb{P}^5$.

Remark 1.2. Although the proof is different, this example is also correct if char $k = 2$. Mohan-Kumar supplied a proof for Luis Aguirre to use in his proof in his 2019 PhD thesis.

We see in this example that the singular conics in families (B) and (C) can be deformed in the parameter space $\mathbb{P}^5$ to the smooth conics in family (A), so they are *smoothable*. In general, given a variety or a family of varieties, it is useful to know when the general member is smooth, as occurred in Example 1.1. If not, can one modify the general singular member to obtain a smooth variety? These are the kinds of questions the smoothing theorems attempt to answer. Here's a brief outline:

2. Singularities in algebraic geometry.
3. Theorems of Hironaka and Bertini, degeneracy loci of maps of vector bundles.
4. Recent work with Rao.

2. Singularities in algebraic geometry

In this section we define singular and smooth points on an algebraic variety, illustrating with various examples.

2.1. **Affine varieties.** For a field k, we define *affine n-space* $\mathbb{A}^n_k$ to be the set k^n with the Zariski topology, meaning that the closed sets are the common zero locus of a family of polyonomials $f_\alpha \in k[x_1, \dots, x_n]$, i.e. the closed sets $Z \subset \mathbb{A}^n$ are

$$Z(f_\alpha) = \{\overline{a} \in \mathbb{A}^n : f_\alpha(\overline{a}) = 0 \text{ for all } \alpha\}.$$

Given a closed set $Z \subset \mathbb{A}^n$, we define $I_Z = \{f \in k[x_1, \dots, x_n] : f(\overline{a}) = 0 \text{ for all } \overline{a} \in Z\}$. It is easy to show that I_Z is an ideal. If $k = \overline{k}$ is algebraically closed, Hilbert's Nullstellensatz tells us that $Z(I_Z) = Z$, so that we may take the f_α from the ideal I_Z. Furthermore Hilbert's basis theorem says that every ideal in $k[x_1, \dots, x_n]$ is finitely generated, so that we may take the f_α from a **finite** generating set for I_Z. The closed set $Z \subset \mathbb{A}^n$ is an *affine variety* if I_Z is a prime ideal. Recall that an ideal $P \subset k[x_1, \ldots, x_n]$ is *prime* if $fg \in P \Rightarrow f \in P$ or $g \in P$.

Example 2.1. In $\mathbb{A}^2$ with coordinates x, y, the ideals $(y - x^2)$ and $(xy - 1)$ are prime and give affine conic varieties, while the ideal (xy) is not prime. Looking at Example 1.1, you might wonder about the non-prime ideal (x^2). It doesn't show up in this context, because $Z(x^2) = Z(x)$, so when we take the ideal defined by closed set, we get the prime ideal (x). On the other hand, (x^2) defines a *closed subscheme* $Z \subset \mathbb{A}^2$ in the language of Grothendieck's scheme theory. Topologically Z is the same as $Z(x)$, but has a different structure sheaf of regular functions.

2.2. **Singular points on affine varieties.** We begin with an example for motivation.

Example 2.2. Consider the variety $X \subset \mathbb{A}^3_{\mathbb{R}}$ defined by the equation $f = x^2+y^2-z^2 = 0$, the quadric cone. If we asked our Calculus III students about tangent planes to V, they would take a point $\overline{a} = (a_1, a_2, a_3) \in V$ and write down the equation

$$\nabla f(\overline{a}) \cdot (\overline{x} - \overline{a}) = 0.$$

This works well enough except when $\overline{a} = (0, 0, 0)$, when their "tangent plane" would turn out to be all of $\mathbb{R}^3$ because all the partials of f vanish there. A point on a variety should be smooth or nonsingular if it's tangent space has the right dimension.

The story over an arbitrary algebraically closed field k is the same. If $X \subset \mathbb{A}^n_k$ is defined by equations f_j, the tangent space to X at $\overline{a}$ is defined by

$$T_{X,\overline{a}} = \{\overline{x} \in \mathbb{A}^n : \sum_{i=1}^{n} \partial f_j/\partial x_i(\overline{a})(x_i - a_i) = 0 \text{ for all } f_j \in I_V\}$$

and $\overline{a}$ is a *nonsingular* or *smooth* point of V if $\dim T_{X,\overline{a}} = \dim X$. Over an algebraically closed field, there is a good theory of dimension for varieties (and Zariski closed subsets), namely $\dim V = \mathrm{tr.deg._k K(X)}$, where $K(X)$ is the function field of X, i.e. the fraction field of the integral domain $k[x_1, \ldots, x_n]/I_X$. Since the tangent space $T_{X,\overline{a}}$ is defined by linear equations whose coefficients come from the derivative matrix $(\partial f_j/\partial x_i)$, the dimension of $T_{X,\overline{a}}$ is $n - \mathrm{rank}\ (\partial f_j/\partial x_i)(\overline{a})$. Thus $\overline{a} \in X$ is nonsingular point if

$$n - \dim X = \mathrm{rank}(\partial f_j/\partial x_i)(\overline{a}).$$

For $r \in \mathbb{Z}$, the set of points $\{\overline{a} : \mathrm{rank}(\partial f_j/\partial x_i)(\overline{a}) < r\}$ is a Zariski closed set, because it is defined by the vanishing of the $r \times r$ minors, which are polynomial equations. It follows that the locus of smooth points in X is Zariski-open in X, in other words, the set of singular points $\operatorname{Sing} X \subset X$ is a Zarsiki closed subset.

Theorem 2.3. *If $X \subset \mathbb{A}^n$ is an algebraic variety over $k = \overline{k}$, then* $\operatorname{Sing} X$ *is a proper closed subset of X, so X has a dense open subset of nonsingular points.*

Proof. The idea is to show that X is birational to a hypersurface in $\mathbb{A}^{\dim X+1}$ to reduce to the case when X is defined by a single equation $f = 0$. Then if $\operatorname{Sing} X = X$, the partials $\partial f/\partial x_i$ lie in (f), contradicting the degrees. □

Example 2.4. A few plane curve singularities at the origin for $X \subset \mathbb{A}^2$. See figure 1.

(a) $xy = x^6 + y^6$ defines a node.
(b) $x^3 = y^2 + x^4 + y^4$ defines a cusp.
(c) $x^2 = x^4 + y^4$ defines a tacnode.
(d) $x^2y = xy^2 + y^4$ defines a triple point.

Example 2.5. We look examine the cusp $Z = Z(x^2 - y^3) \subset \mathbb{C}^2$, following Milnor [2] and Mumford [3, p. 13]. Let $B = \{(x, y) : |x|^2 + |y|^2 \leq 1$ be the closed unit ball about the origin in $\mathbb{C}^2 \cong \mathbb{R}^4$. Then $\partial B \cong S^3$ meets Z transversely in a one dimensional manifold, i.e. $Z \cap \partial B$ is a real curve on S^3, a possibly interesting knot or link. For $t \in \mathbb{C}$, the complex line $x = ty$ intersects the curve $x^2 = y^3$ in three points: substitution gives $t^2y^2 = y^3$ which has a double root $y = 0$ and simple root $y = t^2$, so that $x = t^3$. Thus each point on Z is uniquely written $x = t^3, y = t^2$ for $t \in \mathbb{C}$. Now $(x, y) = (t^3, t^2) \in \partial B$ iff $|t|^6 + |t|^4 = 1$. If λ is the unique positive solution to $\lambda^6 + \lambda^4 = 1$, then

$$Z \cap \partial B = \{(\lambda^3 e^{3i\theta}, \lambda^2 e^{2i\theta}) : 0 \leq \theta \leq 2\pi\}.$$

Observe that Z lies on the torus $T \subset \partial B$ given by

$$T = \{(x, y) : |x| = \lambda^3, |y| = \lambda^2\} \subset \partial B \cong S^3$$

and we see that $Z \cap \partial B$ is the torus knot corresponding to the rational slope 3/2, so it is the trefoil knot. See figure 2. It appears that the singularity $x^r - y^s$ could be treated similarly for $(r, s) = 1$. It might be interesting to determine exactly which knots and links arise from singularities of complex plane curves. The node $xy = 0$ gives not a knot, but a link.

References

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[3] D. Mumford, *Algebraic Geometry I, Complex projective varieties* Grundlehren der mathematischen Wissenschaften 221, Berlin-Heidelberg-New York: Springer-Verlag, 1976.