DIAGONALIZATION OF MATRICES OF CONTINUOUS FUNCTIONS

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Definition 1. *A* (complex) algebra *is a ring* $(\mathcal{A}, +, \cdot)$ *along with a scalar multiplication* $\cdot : \mathbb{C} \times \mathcal{A} \longrightarrow \mathcal{A}$ *that makes* $\mathcal{A}$ *into a vector space and such that* $\lambda \cdot (AB) = (\lambda \cdot A)B = A(\lambda \cdot B)$ *for all* λ *in* $\mathbb{C}$ *and* A *and* B *in* $\mathcal{A}$.

Throughout these notes, we will assume that our algebras $\mathcal{A}$ possesses a unit; i.e., a multiplicative identity.

Example 1. *For each natural number* n*, the set* $\mathrm{M}(n, \mathbb{C})$ *of* $n \times n$ *matrices is a ring under matrix addition and multiplication. Scalar multiplication of a matrix* A *by a complex number* λ *is defined by multiplying each entry of* A *by* λ.

Example 2. *Let* X *be a compact Hausdorff space. Let* $C(X)$ *denote the set of continuous complex valued functions on* X*. Pointwise addition, multiplication, and scalar multiplication make* $C(X)$ *into an algebra.*

Note that if X consists of a single point, $C(X)$ is (isomorphic to) $\mathbb{C}$.
We can combine Examples 1 and 2:

Example 3. *For each compact Hausdorff space* X *and each natural number* n*, the set* $\mathrm{M}(n, C(X))$ *of* $n \times n$ *matrices with entries in* $C(X)$ *forms an algebra.*

Definition 2. *Let* A *and* B *be elements of an algebra* $\mathcal{A}$*. We say that* A *and* B *are* similar *if there exists an invertible element* S *of* $\mathcal{A}$ *such that* $B = SAS^{-1}$.

It is easy to show that similarity is an equivalence relation.

Definition 3. *A* $*$-algebra *is an algebra* $\mathcal{A}$ *equipped with an involution; that is, a map* $* : \mathcal{A} \longrightarrow \mathcal{A}$ *with the property that* $(A^*)^* = A$ *for all* A *in* $\mathcal{A}$.

In Example 1, the involution is complex conjugate transpose. In Example 2, the involution is pointwise complex conjugation; i.e., $f^*(x) := \overline{f(x)}$ for every x in X. The involution in Example 4 is a combination of the involutions in Examples 2 and 3:

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{pmatrix}^* = \begin{pmatrix} \overline{f_{11}} & \overline{f_{21}} & \overline{f_{31}} & \cdots & \overline{f_{n1}} \\ \overline{f_{12}} & \overline{f_{22}} & \overline{f_{32}} & \cdots & \overline{f_{n2}} \\ \overline{f_{13}} & \overline{f_{23}} & \overline{f_{33}} & \cdots & \overline{f_{n3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{f_{1n}} & \overline{f_{2n}} & \overline{f_{3n}} & \cdots & \overline{f_{nn}} \end{pmatrix}$$

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Definition 4. *Let $\mathcal{A}$ be a $*$-algebra. An element U in $\mathcal{A}$ is called* unitary *if U is invertible and $U^{-1} = U^*$. Elements A and B in $\mathcal{A}$ are* unitarily equivalent *if $B = U^*AU$ for some unitary U in $\mathcal{A}$.*

Unitary equivalence is an equivalence relation, and for a $*$-algebra, unitary equivalence implies similarity. However, as we shall see in a minute, the converse is not generally true.

Problem. *Determine the unitary equivalence classes of $\mathrm{M}(n, C(X))$. More specifically, determine a complete set of unitary invariants for $\mathrm{M}(n, C(X))$; i.e., a collection of quantities*

$$f_1(A), f_2(A), \ldots, f_k(A)$$

for each element A of $\mathrm{M}(n, C(X))$ that have the feature that A and B in $\mathrm{M}(n, C(X))$ are unitarily equivalent if and only if $f_i(A) = f_i(B)$ for all $1 \leq i \leq k$.

A reasonable solution to this problem is hopeless in this generality, even if X is a point. Consider elements in $\mathrm{M}(n, \mathbb{C})$ that have the form

$$\begin{pmatrix} 1 & 1 & a_{13} & a_{14} & a_{15} & \cdots & a_{1n} \\ 0 & 2 & 1 & a_{24} & a_{25} & \cdots & a_{2n} \\ 0 & 0 & 3 & 1 & a_{35} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}.$$

By the Jordan Canonical Form Theorem, each of these matrices is similar to the diagonal matrix $\mathrm{diag}(1, 2, 3, \ldots, n)$. However, when $n > 2$, two such matrices are unitarily equivalent if and only if they are equal. Thus there are similarity classes in $\mathrm{M}(n, \mathbb{C})$ that are the union of uncountably many unitary equivalences classes!
The situation is somewhat simpler if we restrict our attention to *normal* matrices.

Definition 5. *An element A in a $*$-algebra is* normal *if $AA^* = A^*A$.*

There are many normal elements in a $*$-algebra; given A in $\mathcal{A}$, the elements $\Re(A) := \frac{1}{2}(A + A^*)$ and $\Im(A) := \frac{1}{2i}(A - A^*)$ are self-adjoint, and therefore normal. Note that $(i\Im(A))^* = -i\Im(A)$, so $i\Im(A)$ is normal; we call such an element *skew-adjoint*. Therefore every element of $\mathcal{A}$ is the sum of two normal elements (in fact, a self-adjoint element and a skew-adjoint one). But of course A itself may or may not be normal.

Proposition 4 (Berberian)**.** *Two normal elements of $\mathrm{M}(n, C(X))$ (in fact, in any C^*-algebra) are unitarily equivalent if and only if they are similar.*

Theorem 5 (Spectral Theorem)**.** *Every normal matrix in $\mathrm{M}(n, \mathbb{C})$ is unitarily equivalent to a diagonal matrix.*

In this case, the unitary invariants are the roots of the characteristic polynomial $P(\lambda) := \det(\lambda I - A)$.
Suppose we have A is in $\mathrm{M}(n, \mathbb{C})$ and that $U^*AU = D$ for some unitary matrix U and some diagonal matrix D. For each $1 \leq i \leq n$, let e_i denote the vector in $\mathbb{C}^n$ that is 1 in the ith slot and 0 elsewhere. Then for each i, we have $De_i = \lambda_i$ for some complex number λ_i, and so $AUe_i = UDe_i = U(\lambda_i)e_i = \lambda_i Ue_i$. In other words, each Ue_i is an eigenvector for A, and the set $\{Ue_i \mid 1 \leq i \leq n\}$ is a complete set of eigenvectors for A. Therefore the columns of U are the eigenvectors of A.

Furthermore, unitary matrices preserve length, so the length of each column of U equals 1. We shall use these observations shortly.

Question. *Let X be a compact Hausdorff space. Is every normal element of $\mathrm{M}(n, C(X))$ unitarily equivalent to a diagonal matrix?*

Example 6. *Define $A \in \mathrm{M}(n, C([-1,1]))$ as follows:*

$$A(x) = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, \quad x \geq 0, \qquad A(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad x < 0.$$

Suppose there exists a unitary matrix U and a diagonal matrix D in $\mathrm{M}(n, C([-1,1]))$ such that $U^*AU = D$. Then $U^*(x)A(x)U(x) = D(x)$ for every $-1 \leq x \leq 1$. For $x > 0$,

$$\det(\lambda I - A(x)) = \det \begin{pmatrix} \lambda - x & -x \\ -x & \lambda - x \end{pmatrix} = (\lambda - x)^2 - x^2 = \lambda^2 - 2\lambda x = \lambda(\lambda - 2x),$$

which has eigenvectors of the form $\begin{pmatrix} f(x) \\ -f(x) \end{pmatrix}$ and $\begin{pmatrix} g(x) \\ g(x) \end{pmatrix}$ for the eigenvalues 0 and $2x$ respectively. Thus for $x \geq 0$, we must have one of the following forms:

$$U(x) = \begin{pmatrix} f(x) & g(x) \\ -f(x) & g(x) \end{pmatrix} \quad \text{or} \quad U(x) = \begin{pmatrix} g(x) & f(x) \\ g(x) & -f(x) \end{pmatrix}, \quad |f(x)| = |g(x)| = \frac{1}{\sqrt{2}}.$$

However, for $x < 0$, we obtain

$$U(x) = \begin{pmatrix} h(x) & 0 \\ 0 & k(x) \end{pmatrix} \quad \text{or} \quad U(x) = \begin{pmatrix} 0 & h(x) \\ k(x) & 0 \end{pmatrix}, \quad |h(x)| = |k(x)| = 1.$$

Clearly, there is no way to choose f, g, h, and k to make U continuous. Thus, while $A(x)$ is diagonalizable for every x in $[-1,1]$, the matrix A itself is not diagonalizable.

In Example 6, notice that the dimension of the eigenspace for 0 jumped at $x = 0$; this is what caused the problem.

Definition 6. *An element A in $\mathrm{M}(n, C(X))$ is* multiplicity-free *if $A(x)$ has n distinct roots for each x in X. Equivalently, A is multiplicity-free if the polynomial $P(x, \lambda) := \det(\lambda I - A(x))$ has n distinct roots for each x in X.*

Is repeated multiplicity the only obstruction to diagonalization?

Definition 7. *A* simple Weierstrass polynomial *of degree n over X is a function $P : X \times \mathbb{C} \longrightarrow \mathbb{C}$ of the form*

$$P(x, \lambda) = \lambda^n + a_{n-1}(x)\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1(x)\lambda + a_0(x),$$

where $a_0, a_1, \ldots a_{n-1}$ are in $C(X)$ and such that for each x, $P(x, \lambda)$ has n distinct roots.

Note that if A is multiplicity-free, its characteristic polynomial is a simple Weierstrass polynomial.

Let P be a simple Weierstrass polynomial over X, and let

$$E := \{(x, \lambda) \in X \times \mathbb{C} \mid P(x, \lambda) = 0\}.$$

Projection onto the first coordinate defines a map $p : E \longrightarrow X$. This map is a covering map, and E is an n-fold cover of X; such a cover is called a *polynomial covering* of X.

Suppose that A in $\mathrm{M}(n, C(X))$ is diagonalizable; choose a unitary U so that U^*AU is diagonal. As we observed earlier, each column of $U(x)$ is an eigenvector for $A(x)$. For each $1 \leq i \leq n$, let $d_i(x)$ be the eigenvalue of $A(x)$ associated to the ith column of $U(x)$. Because U is a continuous function of X, the functions $d_i : X \longrightarrow \mathbb{C}$ are also continuous. As a consequence, we obtain a continuous global factorization of the characteristic polynomial:

$$P(x, \lambda) = \prod_{i=1}^{n} (\lambda - d_i(x)).$$

Example 7. *Let $X = S^1$ and consider the matrix*

$$A(x) = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$P(x, \lambda) = \det(\lambda I - A(x)) = \det \begin{pmatrix} \lambda & -x \\ -1 & \lambda \end{pmatrix} = \lambda^2 - x.$$

As we all know from taking complex analysis, this polynomial does not continuously factor over S^1. Put another way, $P(x, \lambda)$ defines a nontrivial double cover of S^1. Therefore, even though A is normal and multiplicity-free, it does not diagonalize.

If A in $\mathrm{M}(n, C(X))$ is multiplicity-free and its characteristic polynomial globally factors into linear factors, is A diagonalizable?

Definition 8. *A (complex) vector bundle over X is a topological space V and a continuous surjection $p : V \longrightarrow X$ such that*

- *$p^{-1}(x)$ is a complex vector space;*
- *for each x in X, there exists a neighborhood U of X with the property that $p^{-1}(U)$ is homeomorphic to $U \times \mathbb{C}^n$ for some natural number n.*

If V is homeomorphic to $X \times \mathbb{C}^n$ for some natural number n, we say that V is a trivial *vector bundle.*

Definition 9. *An element A of $\mathrm{M}(n, C(X))$ is called a* projection *if $A = A^* = A^2$.*

If A is a projection matrix over X, then the range of A at each point x in X determines a vector bundle $\operatorname{ran} A$ over X. Furthermore, up to vector bundle isomorphism, every vector bundle arises in this way.

Example 8. *Take $X = S^2$, written in cylindrical coordinates (z, θ), and define*

$$A(z, \theta) = \frac{1}{2} \begin{pmatrix} 1 + z & \theta\sqrt{1 - z^2} \\ \bar{\theta}\sqrt{1 - z^2} & 1 - z \end{pmatrix}.$$

Direct computation show that A is a projection and that its characteristic polynomial is $\lambda^2 - \lambda = \lambda(\lambda - 1)$. This polynomial is certainly globally factorable into linear factors. However, the vector bundle $\operatorname{ran} A$ is known to be nontrivial. Interesting side fact: the unit vectors in the range of A at each point determine a topological space homeomorphic to S^3, and via this homeomorphism, we obtain a famous continuous function $p : S^3 \longrightarrow S^2$ called the Hopf vibration.

Thus there are at least three obstructions to diagonalizability of matrices over a compact Hausdorff space X:

- Multiplicities of eigenvalues;
- Existence of nontrivial polynomial covering spaces over X;
- Existence of nontrivial complex vector bundles over X.

It turns out that these are the only obstructions.
If we restrict our attention to multiplicity-free normal matrices, there is a nice result which tells us about diagonalizability in many cases.

Theorem 9 (Grove-Pedersen)**.** *Suppose X is a 2-connected compact CW-complex and that $A \in \mathrm{M}(n, C(X))$ is multiplicity-free. Then A is diagonalizable over X.*

Sketch of proof. The hypothesis $\pi_1(X) = 0$ implies that the characteristic polynomial P of A globally splits into linear factors, so P defines a trivial polynomial cover of X, and we can choose continuously varying eigenspaces Q_i of A. Because we are assuming that A is multiplicity-free, these eigenspaces Q_i are one-dimensional and therefore each determine a *line bundle* $\operatorname{ran} Q_i$ over X. A line bundle L over X is completely determined by its first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$. But $\pi_1(X)$ and $\pi_2(X)$ are both trivial by hypothesis, so $H^2(X, \mathbb{Z}) = 0$ by Hurewicz's theorem. Thus we can pick a continuous non-vanishing section f_i of $\operatorname{ran} Q_i$ for each i. By applying the Gram-Schmidt procedure (which is continuous as a function of X), we may assume that $\{f_1, f_2, \ldots, f_n\}$ are a continuously varying orthonormal frame; the change of basis matrix $U \in \mathrm{M}(n, C(X))$ conjugates A to a diagonal matrix. □

What if A is not multiplicity-free?

Definition 10. *Let X be a topological space. We say that X is σ-*compact *if it is the union of a countable number of compact sets. We say that X is* sub-Stonean *if any two disjoint open σ-compact subsets of X have disjoint closures.*

Example 10. *If X is a countable set with the discrete topology, then X is sub-Stonean.*

Example 11. *Let X be a locally compact σ-compact Hausdorff space and let $\beta(X)$ denote the* Stone-Cěch compactification *of X; roughly speaking, $\beta(X)$ is the largest compact topological space in which X is dense. The topological space $K(X) := \beta(X) - X$ is called the* corona *of X, and with these hypotheses on X, the space $K(X)$ is sub-Stonean.*

Example 12. *No infinite first-countable compact Hausdorff space is sub-Stonean.*

Theorem 13 (Grove-Pedersen)**.** *Suppose X is a compact Hausdorff space with the property that every self-adjoint element of $\mathrm{M}(n, C(X))$ can be diagonalized. Then X is sub-Stonean.*

Proof. Use Urysohn's lemma, along with some cleverness. □

Theorem 14. *Let X be a compact Hausdoff space. The following statements are equivalent:*

- *Every normal matrix over X can be diagonalized.*
- *The topological space X satisfies the following conditions:*
  - ◇ *X is sub-Stonean;*
  - ◇ *the (topological) dimension of X is at most two;*
  - ◇ *every finite cover of every closed subset of X is trivial;*
  - ◇ *every complex line bundle over every closed subset of X is trivial.*

Example 15. *Let Y be a locally compact graph. Then every normal matrix over the corona of Y can be diagonalized if and only if for some compact subset C of Y, the complement $Y - C$ is a countable forest; i.e., a countable disjoint union of trees.*

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