An introduction to the local Langlands program

P. Dillery[1]

[1]Department of Mathematics
University of Maryland, College Park

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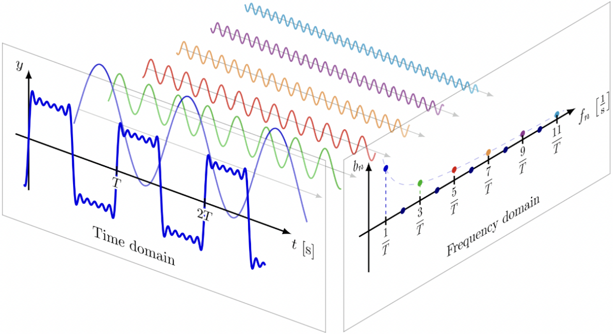

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   - Have Möbius transformations of $\mathbb{H}$: $x + iy \mapsto \frac{ax+biy}{c-diy}$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$

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   - $f(z) = q \prod_n (1 - q^n)^{24}$ $[q = e^{2\pi iz}]$, $k = 12$

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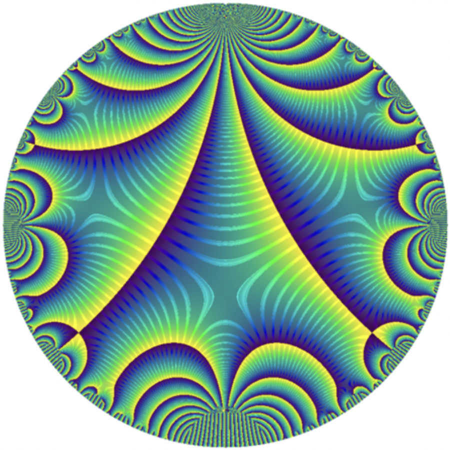
5. Generalization: Functions f on different (higher-dimensional) spaces instead of $\mathbb{H}$ with new symmetries

$f(z) = q\prod_n(1-q^n)^{24}$ $[q = e^{2\pi i z}]$



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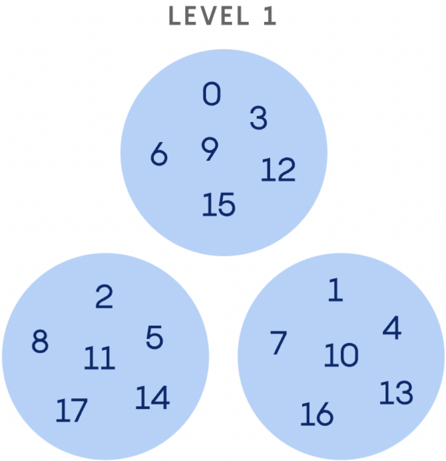
3-adic numbers



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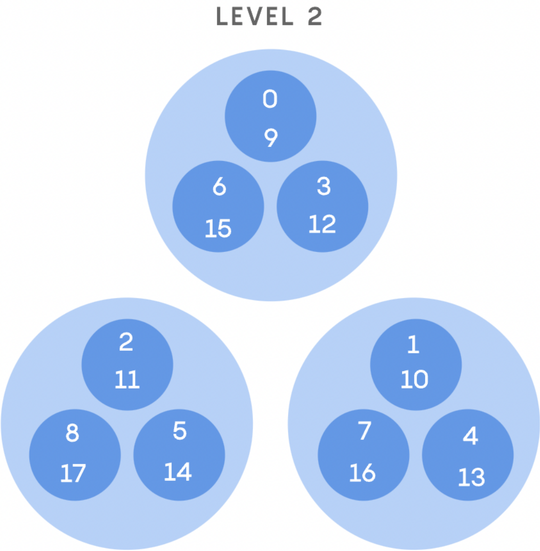

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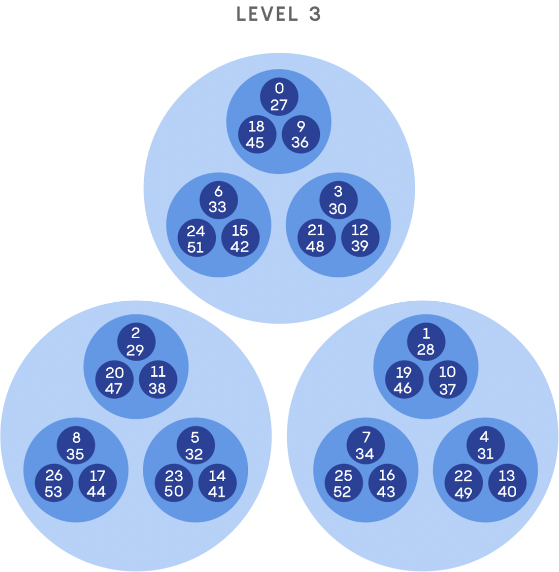

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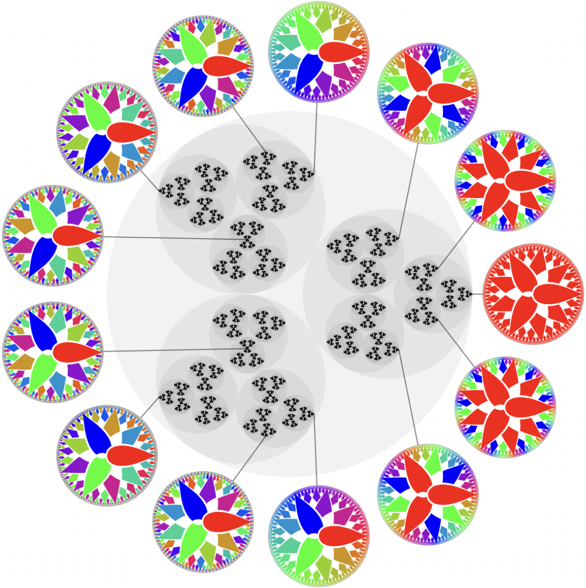

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4. The C_{nv} Groups

C_{2v}	E	C_2	$\sigma_v(xz)$	$\sigma_v'(yz)$		
A_1	1	1	1	1	z	x^2, y^2, z^2
A_2	1	1	-1	-1	R_z	xy
B_1	1	-1	1	-1	x, R_y	xz
B_2	1	-1	-1	1	y, R_x	yz

C_{3v}	E	$2C_3$	$3\sigma_v$		
A_1	1	1	1	z	$x^2 + y^2, z^2$
A_2	1	1	-1	R_z	
E	2	-1	0	$(x, y)(R_x, R_y)$	$(x^2 - y^2, xy)(xz, yz)$

C_{4v}	E	$2C_4$	C_2	$2\sigma_v$	$2\sigma_d$		
A_1	1	1	1	1	1	z	$x^2 + y^2, z^2$
A_2	1	1	1	-1	-1	R_z	
B_1	1	-1	1	1	-1		$x^2 - y^2$
B_2	1	-1	1	-1	1		xy
E	2	0	-2	0	0	$(x, y)(R_x, R_y)$	(xz, yz)

C_{5v}	E	$2C_5$	$2C_5^2$	$5\sigma_v$		
A_1	1	1	1	1	z	$x^2 + y^2, z^2$
A_2	1	1	1	-1	R_z	
E_1	2	2 cos 72°	2 cos 144°	0	$(x, y)(R_x, R_y)$	(xz, yz)
E_2	2	2 cos 144°	2 cos 72°	0		$(x^2 - y^2, xy)$

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$		
A_1	1	1	1	1	1	1	z	$x^2 + y^2, z^2$
A_2	1	1	1	1	-1	-1	R_z	
B_1	1	-1	1	-1	1	-1		
B_2	1	-1	1	-1	-1	1		
E_1	2	1	-1	-2	0	0	$(x, y)(R_x, R_y)$	(xz, yz)
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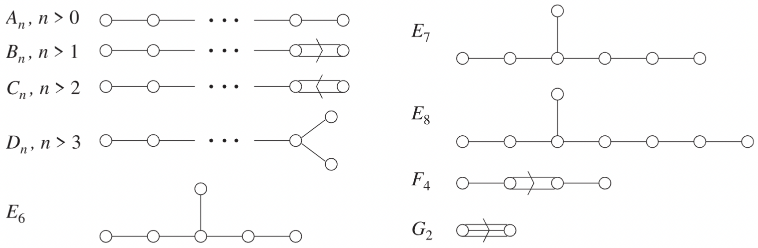

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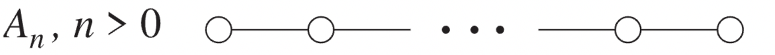

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    - A_n with nontrivial involution: $\mathrm{U}(n+1)$, $\mathrm{SU}(n+1), \dots$

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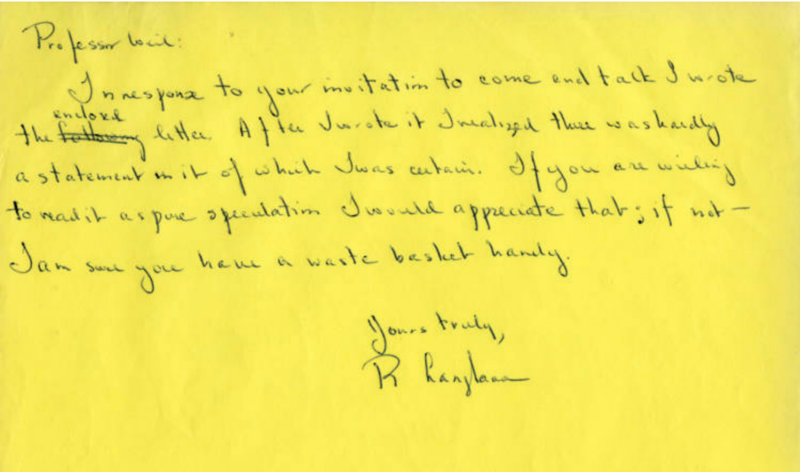
   - 

Professor Weil:

In response to your invitation to come and talk I wrote the ~~following~~ enclosed letter. After I wrote it I realized there was hardly a statement in it of which I was certain. If you are willing to read it as pure speculation I would appreciate that; if not — I am sure you have a waste basket handy.

Yours truly,

R Langlands

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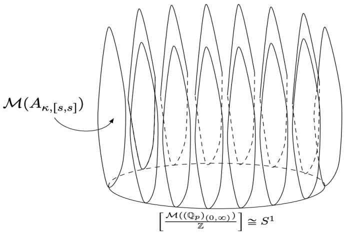

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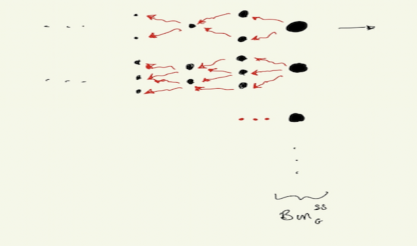

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6. $H^1(\mathcal{E}_{\mathrm{Kal}}, G)$ produces many strange subgroups

Picture sources (in order)

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