DIFFERENTIAL FORM APPROACH TO INTERSECTION HOMOLOGY

1. Preverse Differential Forms

Is there a way to get to intersection homology via differential forms. One standard part of the definition of an n-dimensional stratified space is that every point has a neighborhood homeomorphic to $\mathbb{R}^{n-k} \times cL^{k-1}$, where cL^{k-1} means the cone on a lower dimensional compact stratified space. The chains are

$$I^pC_i(X) = \{\xi \in C_i(X) : \dim(\xi \cap X_{n-k}) \leq \dim(\partial\xi \cap X_{n-k} \leq i - 1 - k - p(k))\},$$

where p is the perversity. Note that

$$\begin{aligned} I^pH_*\left(\mathbb{R}^{n-k} \times cL\right) &= I^pH_*(cL) \\ &= \begin{cases} 0, & * \geq k-1-p(k) \\ I^pH_*(cL - pt) & * < k-1-p(k) \end{cases} \\ &= \begin{cases} 0, & * \geq k-1-p(k) \\ I^pH_*(L \times \mathbb{R}) & * < k-1-p(k) \end{cases} \\ &= \begin{cases} 0, & * \geq k-1-p(k) \\ I^pH_*(L) & * < k-1-p(k) \end{cases} \end{aligned}$$

We are looking for a perverse cochain complex Ω_p^* so that

$$H_c^*\left(X; \Omega_p^*\right) = I^pH_{n-*}^c(X).$$

Motivation: there is a sheaf complex such that in hypercohomology:

$$\mathbb{H}_c^*(X; \mathcal{IS}^*) = I^pH_{n-*}^c(X).$$

We will need to check that our complex has the same local properties. Note that

$$\begin{aligned} H^*(\mathcal{IS}_x^*) &= \mathbb{H}^*\left(\mathbb{R}^{n-k} \times cL; \mathcal{IS}^*\right) \\ &= IH_{n-*}^\infty\left(\mathbb{R}^{n-k} \times cL\right) \\ &= IH_{k-*}^\infty(cL) \\ &= IH_{k-*}(cL, L \times (0,1)) \end{aligned}$$

Using the long exact sequence, you get

$$\begin{aligned} H^*(\mathcal{IS}_x^*) &= IH_{k-*}(cL, L \times (0,1)) \\ &= \begin{cases} 0, & k - * < k - p(k) \\ IH_{k-1-*}(L) & k - * \geq k - p(k) \end{cases} \\ &= \begin{cases} 0, & * > p(k) \\ IH^*(L) & * \leq p(k) \end{cases} \end{aligned}$$

Question: How do we find a complex of differential forms that has this behavior? Two methods:

(1) L^2-cohomology - put appropriate metrics near cone points.
(2) perverse differential forms

We will you use the latter approach. Let X be a Thom-Mather stratified space. (Need tubular neighborhood for each stratum with collapsing conditions.) Key property: these spaces are unfoldable. An **elementary unfolding** is $[0,1] \diagup \text{endpoints} \times L \rightarrow cL$ where $\left\{\frac{1}{2}\right\} \times L$ corresponds to the cone point, and the unfolding is a closed manifold. An **unfolding of** X is a map $\pi : \widetilde{X} \rightarrow X$, where $\widetilde{X}$ is a smooth manifold, and on $\pi^{-1}(X - \Sigma)$, π is a local diffeomorphism, and locally

$$\mathbb{R}^{n-k} \times \widetilde{L} \times \mathbb{R} \rightarrow \mathbb{R}^{n-k} \times cL$$

in distinguished neighborhoods. (Term: unfoldable pseudomanifold.)

A **liftable form** ω on X is a form in $\Omega^*(X - \Sigma)$ such that there exists an $\widetilde{\omega}$ on $\widetilde{X}$ such that $\pi^*\omega = \widetilde{\omega}$ on $\widetilde{X} - \pi^{-1}(\Sigma)$. One can show that

(1) $\widetilde{\omega + \eta} = \widetilde{\omega} + \widetilde{\eta}$
(2) $\widetilde{\omega \wedge \eta} = \widetilde{\omega} \wedge \widetilde{\eta}$
(3) $\widetilde{d\omega} = d\widetilde{\omega}$

Let $\Pi^*(X)$ be the complex of liftable forms. Notice that if Z is an (open) stratum of X, then the projection $\pi : \pi^{-1}(Z) \rightarrow Z$ is a fiber bundle. Given $\eta \in \Omega^*(\pi^{-1}(Z))$, define its vertical degree to be

$$v_Z(\eta) = \min\left\{j \in \mathbb{N} : i_{\xi_0} ... i_{\xi_j}\eta = 0 \text{ for all } \xi_0, ..., \xi_j \text{ tangent to the fibers of } \pi^{-1}(Z) \rightarrow Z.\right\}.$$

If $\omega \in \Pi(X)$, define the perverse degree by

$$\|\omega\|_Z = v_Z\left(\widetilde{\omega}|_Z\right).$$

We now define

$$\Omega_p^*(X) = \left\{\omega \in \Pi^*(X) : \|\omega\|_{X_{n-k}}, \|d\omega\|_{X_{n-k}} \leq p(k)\right\}.$$

The claim is that

$$H^*\left(\Omega_p^*\left(\mathbb{R}^{n-k} \times cL\right)\right) = \left\{\begin{array}{cc} 0, & * > p(k) \\ H^*\left(\Omega_p^*(L)\right) & * \leq p(k) \end{array}\right. .$$

Corollary: $H^*\left(\Omega_p^*(X)\right) = I^pH_{n-*}^{\infty}(X)$.

Idea of proof: We must show

(1) $H^*\left(\Omega_p^*(L \times \mathbb{R})\right) = H^*\left(\Omega_p^*(L)\right)$
(2) $H^*\left(\Omega_p^*(cL)\right) = \left\{\begin{array}{cc} 0, & * > p(k) \\ H^*\left(\Omega_p^*(L)\right) & * \leq p(k) \end{array}\right.$

Proof of (1): standard Bott-Tu proof. With $i : L \rightarrow L \times \mathbb{R}$, $pr : L \times \mathbb{R} \rightarrow L$, both i^* and pr^* preserve Ω_p^*. Note that $pri = id$, so $\pm(ipr - id) = dH - Hd$, where $H = \int_{t_0}^t dt$ on forms $\alpha \wedge dt$ and zero otherwise. Then one gets a chain homotopy from $\Omega_p^*(L)$ to $\Omega_p^*(L \times \mathbb{R})$.

Proof of (2): Special case where L is a manifold: can get a chain homotopy between $L \times \{0\}$ and $\Pi^*(cL)$, and one between $\Omega_p^*(cL)$ and $\Omega_p^*(L \times 0)$.

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