Are Deligne-Lusztig representations Deligne-Lusztig?

Except when they are complex?

1. Representation theory

Let K be a (connected) compact Lie group, and let π an irreducible representation of K. This means that π is finite-dimensional, i.e. π is a homomorphism from K to $GL(V)$, where V is a finite dimensional (complex) vector space. It makes sense to talk about $\Theta_\pi(h) = Tr(\pi(h))$, a scalar-valued function. It turns out that this characterizes the representation π up to unitary equivalence, ie. if π' is another irreducible representation on a space V', and it $\Theta_{\pi'} = \Theta_\pi$ then there is an isomorphism $T : V \to V'$ such that $\pi'(g) \circ T = T \circ \pi(g)$ for all $g \in K$. We call Θ_π is called the **character** of π. This function is a class function, ie. it is constant on conjugacy classes. However it need not be multiplicative, so it is not necessarily a homomorphism. We have in general

$$\Theta_\pi(k)\,\Theta_\pi(k') = \deg(\pi)\int_K \Theta_\pi\left(kuk'u^{-1}\right) du.$$

So, we understand Θ_π, hence π, if we can evaluate Θ_π on a set of conjugacy class representatives. If K contains a maximal torus T (a subgroup isomorphic to a product of S^1s), then every element of K lies in some conjugate of this torus. So if, we understand which functions on T are characters, then we understand the representation theory.

Examples:

- $K = SO_2(\diagdown)$
$$\begin{aligned} &= \left\{G \in M_2(\mathbb{R}) : GG^T = 1 \text{ and } \det G = 1\right\} \\ &= S^1 = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\} \leftrightarrow e^{\iota\theta} \end{aligned}$$

  This has only one-dimensional irreducible representations, which are $z \in S^1 \mapsto z^n$ for fixed n in $\mathbb{Z}$. In terms of actions, K acts on $V = \mathbb{C}$ by letting $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ act by multiplication by $e^{in\theta}$. Then $\Theta_\pi(z) = z^n$ in this case.
- $K = SU_2(\mathbb{C}) = \{g \in M_2(\mathbb{C}) : \det g = 1 \text{ and } g^*g = 1\}$, which is isomorphic to $SO_3(\mathbb{R})$, which is topologically $\mathbb{RP}^3$. It is not abelian. A maximal torus looks like $T = \left\{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in S^1\right\} \cong$

S^1. The irreducible representations of $SU_2(\mathbb{C})$ are realized in $\mathbb{C}[x,y]_{\text{hom}} = \{\text{homogeneous polynomials in 2 variables}\}$. Specifically, we act by transposition:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f(x,y) = f(ax+by, cx+dy).$$

If you fix the degree d to get V_d, then this is a $(d+1)$-dimensional irreducible representation (basis $x^d, x^{d-1}y, ..., y^d$). The matrix in this basis is a diagonal matrix with z^{-i+j} along the diagonal. So

$$\begin{aligned} \Theta_\pi(z) &= z^{-d} + z^{-d+2} + ... + z^d \\ &= \frac{z^{d+1} - z^{-(d+1)}}{z - z^{-1}} \end{aligned}$$

This character is a sum of terms of various weights, and the highest weight is d. The denominator is the difference of the eigenvalues, the Weyl denominator. These are class functions on $SU_2(\mathbb{C})$.

Next time: Let's 'geometrize' in the 'algebro' sense, working with algebraic groups over $\mathbb{C}$ and use Weyl's unitary trick.

Complexifying: $SU_2(\mathbb{C}) \mapsto SL_2()$, because $\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$.

1.1. **Some questions that came up.** What's an irreducible representation? A representation is a linear action of a group G on a (complex) vector space V. That is, it is a homomorphism from G to $GL(V)$. If G is finite-dimensional, we can think of the codomain and range as consisting of matrices. Two bad examples:

(1) The rotation representation of $SO_2(\mathbb{R})$ on $\mathbb{R}^2$.
(2) The shear representation of $\mathbb{R}$ on $\mathbb{R}^2$.

On the level of actions, the second is $t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ty \\ y \end{pmatrix}$.

These are bad in different ways:

The rotation representation has no simultaneous eigenvectors, i.e. there no vector v such that $gv = \lambda(g)v$ for all $g \in G$, so it's irreducible. But there are complex simultaneous eigenvectors. Specifically,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = e^{\pm i\theta}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}.$$

We say this representation is **irreducible** but not **absolutely irreducible** because we make it reducible by changing scalars. This isn't a problem if the field is algebraically closed. Also if the characteristic is zero, it also avoids other problems. This why we work over $\mathbb{C}$. On the level of matrices,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

isn't diagonalizable over $\mathbb{R}$. But

$$\begin{aligned} & Int\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) \\ = & \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}. \end{aligned}$$

On the other hand, the shear representation has an invariant vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; but only one (up to scaling). Even passing to $\mathbb{C}$ doesn't give more eigenvalues. Unlike the rotation representation, where we could write

$$\mathbb{C}^2 = \mathbb{C}\begin{pmatrix} 1 \\ i \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 \\ -i \end{pmatrix},$$

a decomposition that is invariant under rotation (**completely reducible case)**, there is no way to write in terms of the shear representation

$$\mathbb{C}^2 = \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \text{something invariant},$$

so we say that this action is **reducible but indecomposable**. On the level of matrices, the real number t acts by the matrix

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

it's unipotent. In a crude sense, **reducing** a representation means conjugating it so all the matrices are block upper triangular. **Decomposing** a representation means conjugating it so all the matrices are block diagonal. The representation is irreducible, respectively indecomposable, if this cannot be done nontrivially.

Unipotence causes reducibility but indecomposability. We try to avoid it if we want to reduce the study of represention theory to the study of irreducible representations.

Good news: This never happens for complex representations of compact groups. Any representation of a compact group is completely reducible.

Some words:

linearly reductive: irreducibility of representations

reductive: how many unipotents? (i.e. few)

In characteristic zero, both notions are the same.

Eg - additive group of real numbers in not reductive.

Last time, we discussed examples of representations of compact connected Lie groups, and I mentioned Weyl's unitarian trick: every irreducible representation of a connected compact Lie group is the restriction of a unique holomorphic representation of an associated complex Lie group. Good: then you can do algebraic geometry on complex Lie groups.

Example: $K = SO_2(\mathbb{R}) = \left\{ g \in GL_2(\mathbb{R}) : \det(g) = 1, g^T g = 1 \right\}$ is isomorphic to S^1 by $e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. It's irreducible representations one-dimensional and are $z \mapsto z^n$. The complex picture: $G = K^{\mathbb{C}} = SO_2(\mathbb{C})$. (Why is this the right answer? Algebraic groups ... On the level of Lie algebras, $\mathfrak{so}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{so}_2(\mathbb{C})$, but this only identifies the complex group up to covers.

$$\mathfrak{so}_2(\mathbb{R}) = \left\{ X \in \mathfrak{gl}_2(\mathbb{R}) : \mathrm{tr} X = 0,\ X + X^T = 0 \right\}.$$

Then $SO_2(\mathbb{C})$ is abelian, and it's isomorphic to $\mathbb{C}^\times$. $z \mapsto \begin{pmatrix} \frac{1}{2}(z + z^{-1}) & \frac{1}{2i}(z + z^{-1}) \\ -\frac{1}{2i}(z + z^{-1}) & \frac{1}{2}(z + z^{-1}) \end{pmatrix}$ (a rational isomorphism). We call $\mathbb{C}^\times$ an algebraic torus. What do the representations of $\mathbb{C}^\times$ look like? The representation $z \mapsto z^n$ is the restriction of the representation $z \mapsto z^n$ of $\mathbb{C}^\times$. These are all the holomorphic representations of $\mathbb{C}^\times$. but there are others (eg $z \mapsto z^n \overline{z}^m$).

Example: $K = SU_2(\mathbb{C}) = \{ g \in GL_2(\mathbb{C}) : \det g = 1, gg^* = 1 \}$. Note that $\mathfrak{su}_2(\mathbb{C}) = \{ X \in \mathfrak{gl}_2(\mathbb{C}) : \mathrm{tr} X = 0,\ X + X^* = 0 \}$. This is a real, not complex, vector space. Why not just complexify? $\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra – of what? This is $\mathfrak{sl}_2(\mathbb{C}) = \{ X \in \mathfrak{gl}_2(\mathbb{R}) : \mathrm{tr} X = 0 \}$. Ruth: the isomorphism from $\mathfrak{sl}_2(\mathbb{C})$ to $\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ is $X \mapsto \frac{1}{2}(X + iX) + \frac{1}{2i}(X - iX)\, i$.

Last time, we talked about complexification. This is a way of passing from (compact) connected (real) Lie groups to complex Lie groups. As follows, given K a compact connected Lie group, its (real) Lie algebra is $\mathfrak{k}$, then let $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra. For K compact, there exists a connected complex Lie group G whose Lie algebra is $\mathfrak{g}$. But Lie algebras don't see covers; so we choose G so that $\pi_1 G = \pi_1 K$. Then G is the complexification of K. This obeys a universal property.

$$\begin{array}{ccc} K & \rightarrow & H \quad \text{(complex Lie group)} \\ \downarrow \text{(smooth)} & \nearrow \text{(analytic)} & \\ G & & \end{array}$$

But, for noncompact Lie groups, $K \to G$ may have a kernel. For example, if $K = SO_2(\mathbb{R}) = \left\{g \in GL_2(\mathbb{R}) : gg^T = 1\right\}$, then $\mathfrak{k} = \mathfrak{so}_2(\mathbb{R}) = \left\{X : X + X^T = 0\right\} \cong \mathbb{R}$. Then

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{so}_2(\mathbb{C}) \\ &= \left\{X \in \mathfrak{gl}_2(\mathbb{C}) : X + X^T = 0\right\} \cong \mathbb{C} \end{aligned}$$

Among all the complex Lie groups with $Lie(G) = \mathbb{C}$, we pick the one with $\pi_1(G) = \mathbb{Z}$, which is $G = \mathbb{C}^\times$. More explicitly,

$$\begin{aligned} G &= SO_2(\mathbb{C}) \\ &= \left\{g \in GL_2(\mathbb{C}) : gg^T = 1, \det g = 1\right\} \end{aligned}$$

Also,

$$\begin{aligned} K &= SU_2(\mathbb{C}) = SU_{2,\mathbb{C}/\mathbb{R}}(\mathbb{R}) \\ &= \left\{g \in GL_2(\mathbb{C}) : \det(g) = 1, gg^T = 1\right\} \\ \mathfrak{k} &= \mathfrak{su}_2(\mathbb{C}) \end{aligned}$$

Last time, $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$. Among the complex connected Lie groups G with Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we pick the one with $\pi_1 G = \{1\}$, namely $G = SL_2(\mathbb{C})$ (not $PGL_2(\mathbb{C})$).

Last time: Weyl's unitarian trick: Every continuous irreducible representation of K (must be compact), i.e,. a continuous homomorphism from K to $GL_n(\mathbb{C})$, extends uniquely to an analytic representation of $G = K^{\mathbb{C}}$, i.e. an analytic homomorphism from G to $GL_n(\mathbb{C})$. For example, irreducible continuous representations of $SO_2(\mathbb{R}) \cong S^1$ are of the form $z \mapsto z^n$. Irreducible analytic representations of $SO_2(\mathbb{C}) \cong \mathbb{C}^\times$ are of the form $z \mapsto z^n$ (always 1-dimensional). Next example: Two special irreducible continuous representations of $SU_2(\mathbb{R})$:

- The 2-dim representation coming from $K \hookrightarrow GL_2(\mathbb{C})$.
- The 3-dimensional adjoint representation: K acts on $\mathfrak{su}_2(\mathbb{C})$ byconjugation, and $\mathfrak{su}_2(\mathbb{C})$ is 3-dimensional. The action preserves the Killing form:

$$\begin{aligned} X \otimes_{\mathbb{R}} Y &\mapsto \operatorname{tr}([X, [Y, \bullet]] : \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{su}_2(\mathbb{C})) \\ &= \operatorname{tr}(ad(X)\, ad(Y)). \end{aligned}$$

The Killing form of a semi-simple Lie algebra is negative definite if and only if the corresponding Lie group is compact. Note that semisimple Lie groups have finite fundamental group. So this means that we have a map to the orthogonal group.

$$\begin{aligned} SU_2(\mathbb{C}) &\to 0 \text{ (Killing form)} \\ &= O(3,0) = O(0,3) = O_3(\mathbb{R}). \end{aligned}$$

By connectedness, it maps to $SO_3(\mathbb{R})$. By Lie algebras, it is surjective, and it has a kernel (the kernel of the adjoint representation is the center of the group) $\mathbb{Z}_2$ (generated by -1). Thus,

$$\begin{aligned} K/\pm 1 &\cong SO_3(\mathbb{R}) \\ \text{so } K &= Spin_3(\mathbb{R}). \end{aligned}$$

The $(d+1)$-dimensional representation is $sym^d(\mathbb{C}^2)$ (there's exactly one in each dimension).

These representations extend to $SL_2(\mathbb{C})$, as follows. The 2-d representation extends in the obvious way, with $SL_2(\mathbb{C}) \hookrightarrow GL_2(\mathbb{C})$. All others extend as symmetric powers. The three-dimensional representation is the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$.

2. Representations of Complex Groups

Classical examples of these: SL_n (simple), $GL_n = SL_n \times GL_1$ (reductive). SO_n, Sp_{2n}. We saw that every connected, compact Lie group K has an associated connected complex Lie group $G = K^{\mathbb{C}}$, and the restriction map from LHS=holomorphic irreducible representations of G to RHS=continuous irreducible representations of K is a bijection. Note that on the RHS, we have a nice structure theory: every conjugacy class in K intersects a fixed but arbitrary maximal connected abelian subgroup called a torus T (because it is a torus $\cong (S^1)^r$ as a Lie group, with r =rank). On the LHS, this is not true. We still have nice maximal connected abelian subgroups, but they don't meet every conjugacy class. For complex groups, a maximal connected, abelian, (consists of semisimple – diagonalizable – elements) is called an (algebraic) torus A. As a Lie group, $A \cong (\mathbb{C}^\times)^r$ for some r. For example, in $SU_2(\mathbb{C})$, $T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in S^1 \right\}$. In $G = K^{\mathbb{C}} = SL_2(\mathbb{C})$, $A = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^\times \right\}$ but not $\left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\} = U$, which is a maximal, connected, abelian subgroup of G but is not an algebraic torus. It turns out that the solution to the conjugacy class problem is to enlarge an algebraic torus to a Borel subgroup. The "definition" is that $B = A \rtimes U$. For example, if $G = SL_2(\mathbb{C})$, $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\}$. Then B consists of upper triangular matrices. We have $B = \{g \in G : g \text{ upper-triangular}\}$. The honest definition from the algebraic point of view is that a Borel subgroup B in G is a maximal connected, solvable subgroup. (Solvable:

$B \geq [B,B] \geq [[B,B],[B,B]] \geq ... \geq 1$.) The geometric definition is that a Borel subgroup B in G is a minimal closed subgroup such that G/B (as a space) is projective (as an algebraic variety). (Topologically: G/B is compact). Borel subgroups are unique up to conjugacy.

Going back to our example, the map $SL_2(\mathbb{C})/B \to \mathbb{CP}^1$ defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} B \mapsto \mathbb{C}\begin{pmatrix} a \\ c \end{pmatrix}$ is a biholomorphism.

Next example: $SL_3(\mathbb{C}) / \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & a^{-1}e^{-1} \end{pmatrix} \right\} \to \mathrm{Flag}_{SL_3} =$ flag variety defined by $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} B \mapsto \left(\mathbb{C}\begin{pmatrix} a \\ d \\ g \end{pmatrix}, \mathbb{C}\begin{pmatrix} a \\ d \\ g \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} b \\ e \\ h \end{pmatrix} \right)$. Note that $\mathrm{Flag}_{SL_3} \to Gr(2,1)$.

In general, G/B is called the **flag variety of** G.

Facts:

(1) all Borel subgroups are G-conjugate.
(2) B is self-normalizing, meaning that the normalizer $N_G(B) = B$.

So $G/B \to \{\text{Borel subgroups}\}$ is bijective, with $gB \mapsto gBg^{-1}$.

Another concrete perspective: A (maximal) flag in $\mathbb{C}^n$ is a chain of subspaces $0 \subsetneq V_1 \subsetneq V_2 \subsetneq ... \subsetneq \mathbb{C}^n$, and a Borel in $GL_n(\mathbb{C})$ is the stabilizer of a flag. That is, the stabilizer is $\{g \in GL_n(\mathbb{C}) : gV_i = V_i \ \forall i\}$.

The holomorphic representations of G are realized inside the cohomology of G-equivariant line bundles over G/B. (Borel-Weil-Bott).

3. Borel-Weil-Bott and analogues over other fields

Last time: Let K be a compact, connected Lie group K, and let $G = K^{\mathbb{C}}$ (complex, connected). Inside G is B, the Borel subgroup ($G = \mathrm{Ad}(G) \cdot B$), such that G/B is projective. Obligatory: Consider $Pic(G/B)$, the group of line bundles. Actually, we just consider G-equivariant holomorphic line bundles. In particular, B acts on the fiber over $eB = B$. That is, we have a homomorphism from B to $GL(\text{fiber}) = GL_1(\mathbb{C})$ (abelian), and thus, we get a map from $B/[B,B]$. Recall that $B = A \ltimes U$, where A is an algebraic torus , a maximal subgroup in G sisomorphic to $(\mathbb{C}^{\times})^r$, and U is the unipotent radical, and it happens that $U = [B,B]$. For example, if $K = SU_2(\mathbb{C})$, $G = SL_2(\mathbb{C})$, and B is the block upper triangular, U is $1's$ on the diagonal and upper triangular, A is block.

So, given an equivariant line bundle, we obtain a holomorphic homomorphism $A \to \mathbb{C}^{\times}$ (linear character of A). We can differentiate this homomorphism: $\mathfrak{a} \to \mathbb{C}$. Then $Pic_G(G/B) \to \mathfrak{a}^*$. We can partially invert this map: given an **integral** $\lambda \in \mathfrak{a}^*$, which loosely means "exponentiable" (ie $\exp(\lambda)$ is induced from A to $\mathbb{C}^\times$). Think: $A = \mathbb{C}^\times$, $\mathfrak{a}^* = \mathbb{C}$, and integral means it's in $\mathbb{Z}$. We can now define an action of B on $\mathbb{C}$ via $B \to A \xrightarrow{e^\lambda} \mathbb{C}^\times$. Explicitly,

$$b \cdot z = e^\lambda (a) z,$$

where $b = a \times u$. Then call this $\mathbb{C}_\lambda$. Then $G \times_B \mathbb{C}_\lambda$ is a line bundle over G/B. Its natural G-action makes it eqiuivariant.

With the earlier $K = SU_2, G, B, A$, if $\lambda : \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \mapsto z$, then $G \times_B \mathbb{C}_\lambda$ is the "tautological" bundle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times_B z \mapsto \left(z^{-1} \begin{pmatrix} a \\ c \end{pmatrix}, \mathbb{C} \begin{pmatrix} a \\ c \end{pmatrix} \right).$$

Theorem 1. *(Borel-Weil) If λ is integral and dominant (lies in a certain cone in $\mathfrak{a}^*$), then $\Gamma_{hol}(G/B, \mathcal{L}_{-\lambda})$ is an irreducible G-module. All irreducible G-representations arise in this way. If λ is not dominant, then $\Gamma_{hol}(G/B, \mathcal{L}_{-\lambda}) = 0$.*

Theorem 2. *(Borel-Weil-Bott) If λ is integral and regular (lies in a certain open subset of $\mathfrak{a}^*$ - complexment of hyperplanes), there is an explicit integer i_0 such that*

$$H^i (G/B, \mathcal{L}_{-\lambda}) = 0$$

unless $i = i_0$, and $H^{i_0} (G/B, \mathcal{L}_{-\lambda})$ is an irreducible G-module. (Dolbeault cohomology)

In our example, $\mathfrak{a}^* = \mathbb{C}$ (with $\lambda \leftrightarrow 1$), integral means $\mathbb{Z}$, dominant means positive, regular means nonzero. $\mathcal{L}_{-\lambda}$ is the dual of the tautological bundle, which has a two-dimensional space of global holomorphic sections. We are looking for holomorphic maps $G \to G \times_B \mathbb{C}_{-\lambda}$. We'll look instead at maps $f : G \to \mathbb{C}$ such that $f(gau) = e^\lambda (a) f(g)$. Given f, can construct $F : G \to G \times_B \mathbb{C}_{-\lambda}$ by $g \mapsto g \times_B f(g)$. And, given $F : G \to G \times_B \mathbb{C}_{-\lambda}$, we may define $f : G \to \mathbb{C}$ by $f(g) = z$, where $F(g) = g \times_B z$.

We have the Cartan decomposition $G = HB$, where B is as before and $H = SO_2(\mathbb{C})$, by Gram-Schmidt orthogonalization. One can check that $H \cap B = \{\pm 1\}$. The map $f : G \to \mathbb{C}$ as above is determined by its restriction to H; nearly any map $H \to \mathbb{C}$ can be extended to a suitable map $G \to \mathbb{C}$. (Nearly: up to ± 1) However, the extension is not automatically holomorphic.

Explicit calculation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{a^2+c^2}} & -\frac{c}{\sqrt{a^2+c^2}} \\ \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \end{pmatrix} \begin{pmatrix} \sqrt{a^2+c^2} & ? \\ 0 & \frac{1}{\sqrt{a^2+c^2}} \end{pmatrix}$$

So

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{-\lambda} \begin{pmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+c^2}} \end{pmatrix}^{-1} f\left(\begin{pmatrix} \frac{a}{\sqrt{a^2+c^2}} & -\frac{c}{\sqrt{a^2+c^2}} \\ \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \end{pmatrix}\right)$$

So $e^{-\lambda} \begin{pmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+c^2}} \end{pmatrix}^{-1} = \sqrt{a^2+c^2}$. So we want f to pick off either the 11 entry or the 12 entry (and everything else is in the span of those two).

ie $f\begin{pmatrix} a & -c \\ c & a \end{pmatrix} = a$ or c, then the resulting extension is holomorphic. "Obviously", these are the only possibilities up to linear combinations.

Call these two maps X and Y respectively. Then the G-action on $\Gamma(G/B, \mathcal{L}_{-\lambda}) = \mathbb{C}X \oplus \mathbb{C}Y$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY$, and the matrix of this action is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So the character of this representation is $\Theta(g) = \operatorname{tr}(\text{action of } g) = a + d$. In particular, if $g = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, then $\Theta(g) = a + a^{-1} = \frac{a^2 - a^{-2}}{a - a^{-2}}$, where the numerator is the signed sum of linear characters of A, the denominator is the Weyl denominator.