Coarse Algebraic Topology

X - coarse space

X^{q+1} - Cartesian product of $q+1$ copies of X

$\Delta_q(X)$ (or Δ) - diagonal in X^{q+1}

$E \subset X^{q+1}$ is <u>controlled</u> if the coordinate projections $\pi_0, \dots \pi_q$ are pairwise close, & E is <u>bounded</u> if all the coordinate projections are close to a constant map.

Thus every bounded set is controlled, but not conversely in general

Note: When $q=0$, every subset is controlled, & "bounded" ~~coin~~ coincides with its usual meaning.

Lemma: X, Y coarse spaces, $f: X \to Y$ a coarse map.

(a) for every controlled subset $E \subseteq X^{q+1}$, the image

$$f_*(E) = \{ (f(x_0), \dots, f(x_q)) : (x_0, \dots x_q) \in E \}$$

is a controlled subset of Y^{q+1}

(b) for every bounded subset $B \subseteq Y^{q+1}$, its preimage

$$f^{-1}(B) = \{ (x_0, \dots, x_q) : (f(x_0), \dots f(x_q)) \in B \}$$

is a bounded subset of X^{q+1}.

Definition: Let X be a coarse space. A subset D of X^{q+1} is <u>cocontrolled</u> if $D \cap E$ is bounded for every controlled set E.

Remark: For $q=0$, the cocontrolled subsets are precisely ~~that~~ the bounded ones.

Example: Take $X=\mathbb{R}$ with its metric coarse structure, & let $D\subset X^2$ be the union of the 2nd & 4th quadrants; i.e.,

$$D=\{(x,y) : xy\leq 0\}$$

Then D is cocontrolled.

Lemma: X, Y coarse spaces, $f: X\to Y$ a coarse map. If D is cocontrolled in Y^{q+1}, then $f^*(D)$ is cocontrolled in X^{q+1}.

Definition: Let X be a coarse space, G an abelian group. Define

$CX^q(X;G)$ to be the ~~space~~ set of functions $\phi: X^{q+1}\to G$ that have cocontrolled support.

$CX^*(X;G)$ is called the <u>coarse complex of X with coefficients in G</u>.

As in algebraic topology, when $G = \mathbb{Z}$ we suppress it from the notation.

The coboundary map for $CX^*(X;G)$ has the same form as it does in Alexander-Spanier cohomology: i.e.,

$$\partial\phi(x_0, \dots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \phi(x_0, \dots, \hat{x}_i, \dots x_{q+1}).$$

$HX^*(X;G)$ is the <u>coarse cohomology</u> of X

Proposition (for Igor): Coarse cohomology is a contravariant functor from ~~Coa~~ coarse spaces and coarse maps to abelian groups and group homomorphisms.

Proposition: If $f, g : X \to Y$ are close, then

$$f^*, g^* : HX^*(Y;G) \to HX^*(X;G)$$

are equal.

~~Computations~~ Examples:

- Suppose X is bounded; i.e, $X \times X \in \mathcal{E}$. Then
$$HX^q(X;G) = \begin{cases} G & \text{if } q=0 \\ 0 & \text{if } q>0 \end{cases}$$

- For any coarse space X,
$$HX^0(X;G) = \begin{cases} G & \text{if } X \text{ is bounded} \\ 0 & \text{if } X \text{ is not bounded} \end{cases}$$

- If $\mathbb{R}^n$ is given its coarse metric structure, then
$$HX^q(\mathbb{R}^n;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } q=n \\ 0 & \text{if } q\neq n \end{cases}$$

Here is a generator: choose a compactly-supported n-form α with $\int_{\mathbb{R}^n} \alpha = 1$, & let $\Lambda(x_0, \dots x_n)$ denote the oriented n-simplex with vertices $x_0, \dots x_n$. Then

$$\phi(x_0, \dots x_n) = \int_{\Lambda(x_0 \dots x_n)} \alpha$$

determines a coarse n-cocycle that generates ~~HX~~ $HX^n(\mathbb{R}^n;\mathbb{R})$.

Digression: Alexander - Spanier cohomology

X top space, G abelian group

$H^*_c(X;G)$ - Alexander - Spanier cohomology with compact support.

A q-cochain in this theory is represented by an equivalence class of functions

$$z : X^{q+1} \to G$$

that are locally zero on the complement of a compact set; two such functions are equivalent if they agree on a nbhd of the diagonal.

Recall that a coarse structure on a paracompact Hausdorff space is <u>proper</u> if

(i) there is a controlled nbhd of the diagonal;

(ii) every bounded subset of X has compact closure.

for such a coarse structure on X, we have a

<u>character map</u> $c: HX^*(X;G) \to H_c^*(X;G)$.

This map is defined by sending a cocycle ϕ to its truncation to any controlled nbhd of the diagonal.

Remark: If X is a manifold + ϕ is a smooth ~~coeffce~~ cocycle with $\mathbb{R}$ coefficients, then we can take $H_c^q(X)$ as deRham cohomology, + in this case c is the map for which

$$f_0 \otimes f_1 \otimes \cdots \otimes f_q \longmapsto f_0\, df_1 \wedge df_2 \wedge \cdots \wedge df_q$$

for smooth functions $f_0, f_1, \ldots f_q$.

Properties of the character map

Proposition: Let $i: H_c^*(X;G) \to H^*(X;G)$ be the obvious map. Then $i\circ c: HX^q(X;G) \to H^q(X;G)$ is zero for $q \geq 1$.

Proposition: Suppose X is a proper coarse space that is (topologically) path connected. Then

$$c: HX^1(X;G) \to H^1_c(X;G)$$

is injective.

Proof: Let $\phi: X^2 \to G$ be a coarse 1-cocycle & suppose that $c[\phi]$ vanishes in $H^1_c(X;G)$. Then there exists a nbhd $\mathcal{U}$ of the diagonal in $X\times X$ & a compactly (and hence cocontrolledly) supported function g on X such that

$$\phi(x_0,x_1) = g(x_0) - g(x_1) \text{ when } (x_0,x_1)\in\mathcal{U}.$$

Now take $x, x' \in X$ & let γ be a path in X from x to x'. We can choose a sequence of points $x = x_0, x_1, \ldots, x_n = x'$ along γ so that $(x_i, x_{i+1}) \in U$ for all $0 \leq i < n$. Using the cocycle identity for ϕ and the definition of g, we have that

$$\phi(x, x') = \sum_{i=0}^{n-1} \phi(x_i, x_{i+1}) = g(x') - g(x).$$

Therefore $\phi = \partial g$ globally, whence $[\phi] = 0$ in $HX^1(X;G)$.

Proposition: Let X be a locally compact geodesic space (that is, every two points in X can be connected by a geodesic). Then there exists an exact sequence

$$0 \to HX^1(X;G) \xrightarrow{c} H^1_c(X;G) \xrightarrow{i} H^1(X;G).$$

Product Structures

X - coarse space
R - commutative ring

$\phi \in CX^p(X;R)$, $\psi \in CX^q(Y;R)$

$\phi \vee \psi \in CX^{p+q}(X \times Y; R)$ defined by

$$(\phi \vee \psi)\big((x_0,y_0),(x_1,y_1),\dots,(x_{p+q},y_{p+q})\big)$$

$$= \phi(x_0,\dots x_p)\psi(y_p, y_{p+1}\dots y_{p+q})$$

It is straightforward to check that $\phi \vee \psi$ has cocontrolled support, & so this recipe gives us an external product

$$HX^p(X;R) \times HX^q(Y;R) \to HX^{p+q}(X \times Y; R).$$

Question: What happens if we try to define an internal product by composing with the diagonal map $X \to X \times X$?

Answer: This internal product is zero on coarse cohomology

Reason:

$$(\phi \vee \psi)(x_0, \dots x_{p+q}) = \phi(x_0, \dots x_p)\, \psi(x_p \dots x_{p+q})$$

has cocontrolled support even if only one of ϕ, ψ has this property.

Remark: One can define a "secondary" internal product

$$HX^p(X;R) \times HX^q(X;R) \to HX^{p+q-1}(X;R)$$

which is nontrivial.