Proposition: $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent.

Proof: Let $f: \mathbb{Z} \to \mathbb{R}$ be the inclusion map and define $g: \mathbb{R} \to \mathbb{Z}$ by $g(x) = \lfloor x \rfloor$. Then $g \circ f = id_{\mathbb{Z}}$, and $f \circ g(x) = \lfloor x \rfloor$ is close to $id_{\mathbb{R}}$: in fact, we can take $M = 1$.

Remark: We will see later that $\mathbb{R}$ + $\mathbb{R}^2$ are <u>not</u> coarsely equivalent.

Next time

Let Γ be a discrete group, + let S be a set of generators of Γ. For each $\gamma \in \Gamma$, let $|\gamma|$ denote the smallest integer n such that

$$\gamma = s_1 s_2 \cdots s_n, \quad s_i \in S \text{ or } s_i^{-1} \in S \ \forall i.$$

It is easy to see that

$$|\gamma\tau| \leq |\gamma| + |\tau|,$$

from which it follows that $d : \Gamma \times \Gamma \to [0,\infty)$ defined by

$$d(x,y) = |x^{-1}y|$$

is a metric on Γ, called the <u>word metric</u> on Γ associated to the generating set S. Note that d is ~~left-invariant~~ invariant under the left action of Γ on itself by translation:

$$d(\gamma x, \gamma y) = |(\gamma x)^{-1}(\gamma y)| = |x^{-1}y|. = d(x,y)$$

(we could define a right-invariant word metric assoc. to S by $\tilde{d}(x,y) = |xy^{-1}|$).

Clearly, the word metric depends on the choice of generating set S. However, its coarse equivalence class does not; at least in "nice" cases:

Theorem: Let S be a finite generating set for a discrete group Γ, & let d be the associated left-invariant word metric. Suppose $\tilde{d}$ is also a left-invariant metric that is proper (so balls $B_{\tilde{d}}(\gamma, r)$ are finite). Then the identity map $i : (\Gamma, d) \to (\Gamma, \tilde{d})$ is a coarse equivalence. In particular, any two ((left-invariant) word metrics on Γ are coarsely equivalent.

Proof: Let $c = \max\{\tilde{d}(\gamma, e) : \gamma \in S\}$. Then by translation invariance,

$$\tilde{d}(x\gamma, x) \leq c \quad \forall x \in \Gamma.$$

~~By induction & the triangle inequality, we have~~

$$\tilde{d}\cancel{(x, e) \leq c|x|_S} \quad \cancel{\forall x \in \Gamma}$$

Suppose $x = s_1 s_2$, s_i or $s_i^{-1} \in S$, $i = 1, 2$. Then

$$\begin{aligned}\tilde{d}(x,e) &= \tilde{d}(s_1 s_2, e) \\ &\leq \tilde{d}(s_1 s_2, s_1) + \tilde{d}(s_1, e) \\ &= \tilde{d}(s_2, e) + \tilde{d}(s_1, e) \\ &\leq 2c \leq c \cdot |x|_S .\end{aligned}$$

Proceeding by induction, we see that

$$\tilde{d}(x,e) \leq c|x|_S = c \cdot d(e,x) = c \cdot d(x,e). \quad \forall x \in \Gamma;$$

and thus

$$\tilde{d}(x,y) = \tilde{d}(y^{-1}x, e) \leq c \cdot d(y^{-1}x, e) = c \cdot d(x,y) \qquad \forall x, y \in \Gamma,$$

whence $id : (\Gamma, d) \to (\Gamma, \tilde{d})$ is a coarse map.

Going the other way, given $R > 0$, we can find $S > 0$ such that $\tilde{d}(x,e) \leq R \Rightarrow d(x,e) \leq S$ because the ball $B_{\tilde{d}}(e;R)$ is finite. Once again invoking translation invariance shows that $\tilde{id} : (\Gamma, \tilde{d}) \to (\Gamma, d)$ is coarse. ∎

22.5

Definition: The Cayley graph of a discrete group Γ with generating set S is the graph whose vertices are the elements of Γ, with an edge joining $x, y \in \Gamma$ if + only if xy^{-1} or yx^{-1} is in S. Make the Cayley graph into a length space by making each edge isometric to $[0,1]$. Then the word metric on Γ is the restriction of this Cayley graph metric to the vertices of the graph.

Group action terminology:

X - locally compact Hausdorff space

Γ - discrete group, $\Gamma \times X \to X$ group action
$(\gamma, x) \longmapsto \gamma \cdot x$

This action is <u>cocompact</u> if there exists a compact subset K of X such that

$$\Gamma \cdot K := \bigcup_{\gamma \in \Gamma} \gamma \cdot K = X.$$

The action is <u>proper</u> if each $x \in X$ has a nbhd U with the property that $\gamma U \cap U = \phi$ for all but finitely many γ.

Important example: M compact Riem. manifold with universal cover X, $\Gamma = \pi_1(M)$ acting on X via deck transformations.

Proposition: Suppose Γ acts properly and cocompactly by isometries on a connected metric space X. Then X is locally compact + complete, + Γ is finitely generated.

Theorem: (A.S. Švarc, 50s, Milnor, 60s): Let Γ be a group acting properly and cocompactly by isometries on a length space X. Fix a base point x_0 of X, and define $f : \Gamma \to X$ by $f(\gamma) = \gamma \cdot x_0$.

Then f is a coarse equivalence.

Remark: In fact, f is not only a coarse equivalence, but f is also a large-scale Lipschitz equivalence.

Corollary: Let $\tilde{\Gamma}$ be a finite-index subgroup of a finitely generated group Γ. Then the inclusion of $\tilde{\Gamma}$ into Γ is a ~~co~~ coarse equivalence.

Proof: Let X be the Cayley graph of Γ. The theorem implies that $\Gamma \to X$ and $\tilde{\Gamma} \to X$ are both coarse equivalences.

Definition: Groups Γ_1, Γ_2 are <u>commensurable</u> if they have isomorphic finite-index subgroups.

Corollary: If Γ_1, Γ_2 are commensurable, then they are coarsely equivalent.

Converse of this corollary is not true in general, but there are partial converses, such as

Theorem: If Γ is coarsely equivalent to $\mathbb{Z}^n$, then Γ is commensurable with $\mathbb{Z}^n$.

Theorem (Gromov): Let Γ, H be finitely generated ~~Fin~~ discrete groups. Then Γ and H are coarsely equivalent if + only if $\exists$ a locally compact Hausdorff space X admitting commuting cocompact proper actions of Γ and H.