Chapter 2 - Coarse Spaces

Definitions: Let X be a set.

(i) Given $E \subset X \times X$, we let E^{-1} denote the set

$$E^{-1} = \{(\tilde{x}, x) : (x, \tilde{x}) \in E\};$$ we call E^{-1} the <u>inverse</u> of E;

(ii) Given $E', E'' \subset X \times X$, then $E' \circ E''$ denotes the set

$$E' \circ E'' = \left\{ (x', x'') : \exists x \in X \text{ with } \begin{array}{l} (x', x) \in E' \\ (x, x'') \in E'' \end{array} \right\};$$

we call $E' \circ E''$ the <u>product</u> of E' and E''.

Remark: These two operations give the <u>pair groupoid</u> structure on $X \times X$.

If $E = E^{-1}$, we say E is <u>symmetric</u>.

Definition: Let K be a subset of X + E a subset of $X \times X$. Define

$$E[K] = \{x' \in X : (x', x) \in E \text{ for some } x \in K\}.$$

When K is a singleton $\{x\}$, we write

$$E[K] = E_x = \{x' \in X : (x', x) \in E\}$$

$$E^{-1}[K] = E^x = \{x' \in X : (x, x') \in E\}.$$

Recall that a subset A of a top space X is <u>relatively compact</u> if $\overline{A}$ is compact.

Definition: Let X be a top space. A subset $E \subseteq X \times X$ is <u>proper</u> if $E[K]$ and $E^{-1}[K]$ are relatively compact whenever K is relatively compact.

Remark: Note that the inverse of a proper set is proper, + the composition or product of ~~pr~~ two proper sets is proper.

Definition: A <u>coarse structure</u> on a set X is a collection $\mathcal{E}$ of subsets of $X \times X$ such that

- the diagonal is in $\mathcal{E}$;
- if $E \in \mathcal{E}$ and $\tilde{E} \subset E$, then $\tilde{E} \in \mathcal{E}$;
- if $E \in \mathcal{E}$, then $E^{-1} \in \mathcal{E}$;
- if $E', E'' \in \mathcal{E}$, then $E' \circ E'' \in \mathcal{E}$;
- if $E_1, \dots, E_n \in \mathcal{E}$, then $\bigcup_{i=1}^{n} E_i \in \mathcal{E}$ (only finite unions).

The sets in $\mathcal{E}$ are called the <u>controlled sets</u> or <u>entourages</u> for the coarse structure. A set equipped with a coarse structure is called a <u>coarse space</u>.

Examples:

① X a set, $\mathcal{E} = \mathcal{P}(\overset{\text{diagonal}}{\Delta})$. This is the <u>trivial</u> coarse structure on X.

② X a set, $\mathcal{E}$ the collection of all subsets of $X \times X$ with only finitely many points off the diagonal Δ. This is called the <u>discrete</u> coarse structure on X.

③ X a set, $\mathcal{E} = \mathcal{P}(X \times X)$. This is the <u>maximal</u> coarse structure on X.

④ X a top space, $\mathcal{E}$ the collection of all proper subsets of $X \times X$ (see p. 28). This coarse structure is called the <u>indiscrete</u> coarse structure on X. Note that if X is compact, this coincides with the maximal coarse structure.

5 X a metric space, $\mathcal{E}$ be collection of all subsets E of $X \times X$ for which

$$\sup \{ d(x,x') : (x,x') \in E \} < \infty .$$

This is called the <u>bounded</u> coarse structure assoc. to d.

6 Suppose X is equipped with a coarse structure $\mathcal{E}$ + Y is a subset of X. The induced coarse structure on Y ~~consists~~ consists of all subsets of $Y \times Y$ that are in $\mathcal{E}$ when considered as subsets of $X \times X$.

7 Suppose $X, \tilde{X}$ are coarse spaces. The <u>product</u> coarse structure on $X \times \tilde{X}$ consist of all subsets of $(X \times \tilde{X}) \times (X \times \tilde{X})$ that are controlled when projected onto $X \times X$ and $\tilde{X} \times \tilde{X}$.

Definition: A coarse structure on X is <u>connected</u> if each point in $X \times X$ belongs to ~~some~~ some controlled set.

Note that coarse connectedness is rather different from topological connectedness!

The discrete, indiscrete, and maximal coarse structures are always connected. In addition, if X is a metric space in which any two points are a finite distance apart, then the bounded coarse structure is also connected.

Definition: If X has coarse structures $\mathcal{E}$, $\mathcal{F}$ such that $\mathcal{E} \subset \mathcal{F}$, we say $\mathcal{E}$ is <u>finer</u> than $\mathcal{F}$ or $\mathcal{F}$ is <u>coarser</u> than $\mathcal{E}$ (terrible terminology!)

Proposition: Let $\mathcal{S}$ be a family of subsets of $X \times X$. Then there is a unique coarse structure on X that contains $\mathcal{S}$ and is finer than any other coarse structure on X that contains $\mathcal{S}$. We call this coarse structure the coarse structure <u>generated</u> by $\mathcal{S}$.

Proof: Let $\{\mathcal{E}_\alpha\}_{\alpha \in I}$ be the family of all coarse structures on X that contain $\mathcal{S}$; note this collection is nonempty, because $\mathcal{S} \subset \mathcal{P}(X \times X)$, which is a coarse structure. The collection

$$\mathcal{E} := \bigcap_{\alpha \in I} \mathcal{E}_\alpha$$

is a coarse structure on X that obviously contains $\mathcal{S}$ & is finer than any other such coarse structure.

Examples:

1. Let $\mathcal{S}$ be the collection of all one-point subsets of $X \times X$. The coarse structure generated by $\mathcal{S}$ is the finest connected coarse structure on X.

2. Let Γ be a finitely generated group. The bounded coarse structure assoc. to any word metric on Γ is ~~then~~ generated by the sets

$$\Delta_\gamma := \{ (\beta, \beta\gamma) : \beta \in \Gamma \}$$

as γ ranges over Γ (or just a generating set of Γ).

Definition: Let X be a coarse space and let S be a set. Maps $f, \tilde{f} : S \to X$ are <u>close</u> if the set

$$\{ (f(s), \tilde{f}(s)) : s \in S \} \subseteq X \times X$$

is controlled.

Remarks:

- Closeness is an equivalence relation.
- $E \subset X \times X$ is controlled if and only if the two coordinate projections are close. Thus the relation of closeness determines the coarse structure.

Properties of Closeness:

- if $f, \tilde{f} : S \to X$ are close and $g : S' \to S$ is any function, then $f \circ g$ and $\tilde{f} \circ g$ are close.

- if $S = S' \cup S''$ and $f, \tilde{f} : S \to X$ have the feature that their restrictions to S' and S'' are close, then f and $\tilde{f}$ are close.

- any two constant maps are close.

Proposition: Let B be a subset of a coarse space X. TFAE:

(a) $B \times B$ is controlled;

(b) $B \times \{p\}$ is controlled for some $p \in X$;

(c) $B = E_p$ for some controlled set E and some $p \in X$;

(d) the inclusion $B \to X$ is close to a constant map.

A set satisfying these conditions is called <u>bounded</u>.

Examples:

1. If X is a metric space equipped with its bounded coarse structure, then the (coarsely) bounded sets are precisely the d-bounded ones.

2. If X is a top space equipped with the indiscrete coarse structure, then the bounded sets are precisely the sets with compact closure: i.e., the relatively compact sets.