TODA-LLY COOL STUFF

BARBARA SHIPMAN

1. Introduction to the Toda Lattice

Finite non-periodic Toda lattice (Toda, 1967):
Dynamical system: a system of n collinear particles with positions $q_1, ..., q_n$
Repelling force between adjacent particles: $F(r) = e^{-r}$
Potential Energy: $F(r) = -\nabla V(r)$, so $V(r) = e^{-r}$
Total energy:

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{-(q_{i+1}-q_i)}$$

Hamilton's equations:

$$\dot{p}_j = -\frac{\partial H}{\partial q_j},\ \dot{q}_j = \frac{\partial H}{\partial p_j}$$

The resulting equations are:

$$\begin{aligned} \dot{q}_j &= p_j \\ \dot{p}_1 &= -e^{-(q_2-q_1)} \\ \dot{p}_j &= -e^{-(q_{j+1}-q_j)} + e^{-(q_j-q_{j-1})} \\ \dot{p}_n &= e^{-(q_n-q_{n-1})} \end{aligned}$$

It is not clear how to solve this system.

Change of variables and inverse scattering solution (Flaschka, 1972). Recall any equation like $L' = [L, B]$, then the spectrum of L is preserved under this flow. The reason is that $\left(tr\left(L^k\right)\right)' = tr\left(L^k\right)' = tr\left[L^k, B\right]$ (prove by induction), which is zero. (L, B) is called a Lax pair. Let

$$\begin{aligned} a_j &= e^{-(q_{j+1}-q_j)} > 0,\ j = 1...n-1 \\ b_j &= -p_j,\ j = 1...n;\ \sum b_j = c \end{aligned}$$

Let

$$\begin{aligned} X_J &= X = \begin{pmatrix} b_1 & 1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & a_{n-1} & b_n \end{pmatrix} \text{ (Jacobi matrix)}, \\ \Pi_{N_-}X &= \begin{pmatrix} 0 & 0 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ & & a_{n-1} & 0 \end{pmatrix} \end{aligned}$$

Then the Jacobi matrix has distinct, real eigenvalues, and the dynamical system now is

$$\begin{aligned} \dot{X}(t) &= \left[X(t), \Pi_{N_-} X(t)\right], \\ H(t) &= \frac{1}{2} tr\left(X^2\right) \end{aligned}$$

Constants of motion: $H_k = \frac{1}{k} tr\left(X^k\right)$, $k = 2, ..., n$, also $tr(X)$. So it is completely integrable. Note each H_k has its own flow $\dot{X}(t) = \left[X(t), \left(\Pi_{N_-} X(t)\right)^k\right]$. The flows give coordinates on n-dimensional isospectral set within the phase space that is $2n$-dimensional. There are many generalizations of this obtained by extending or changing the phase space on which the Toda lattice is defined. There are many differences in structure of the isospectral sets, some of which are subtle, some of which are qualitatively very different. Many have used these flows to study the topology of the isospectral sets in different versions of the Toda lattice.

A standard inverse scattering scheme can be used to solve these equations. That is, suppose (in general) that we have the matrix equation $\dot{L} = [L, B]$. This is equivalent to the following system of equations:

$$\begin{aligned} L\phi &= \phi\Lambda \\ \dot{\phi} &= -B\phi \end{aligned}$$

where ϕ is the matrix of eigenvectors of L, and Λ is the diagonal matrix of corresponding eigenvalues (which are constant under the flow, since the flow is isospectral). Note that from the first equation and the fact that we have

$$\begin{aligned} \dot{L}\phi + L\dot{\phi} &= \dot{\phi}\Lambda, \text{ or} \\ \dot{L}\phi - LB\phi &= -BL\phi, \text{ so} \\ \dot{L} &= [L, B]. \end{aligned}$$

Note that all steps are reversible, so that these systems are equivalent. Now, suppose we are given the initial matrix $L(0)$, such that $L(0)\phi^0 = \phi^0\Lambda$. Since $L = \phi\Lambda\phi^{-1}$, we can express $B\phi$ entirely in terms of ϕ and Λ. Then you get an IVP for a system of ODEs for $\phi(t)$. You can solve that, and then you can of course obtain $L(t)$.

Symmetric version of Toda lattice:

$$X_s = \begin{pmatrix} b_1 & \alpha_1 & & \\ \alpha_1 & \ddots & \ddots & \\ & \ddots & \ddots & \alpha_{n-1} \\ & & \alpha_{n-1} & b_n \end{pmatrix} = D \begin{pmatrix} b_1 & 1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & a_{n-1} & b_n \end{pmatrix} D^{-1},$$

where $a_i = \alpha_i^2 > 0,\ D = \operatorname{diag}(1, \alpha_1, ..., \alpha_{n-1})$.

New equations:

$$\begin{aligned} \dot{X}_s &= \left[X_s, \operatorname{Skew}(X_s)\right] \\ H_s &= \frac{1}{2} tr\left(X_s^2\right), \end{aligned}$$

where

$$\text{Skew}\,(X_s) = \begin{pmatrix} 0 & -\alpha_1 & & \\ \alpha_1 & \ddots & \ddots & \\ & \ddots & \ddots & -\alpha_{n-1} \\ & & \alpha_{n-1} & 0 \end{pmatrix}$$

There is a 1-1 correspondence between the symmetric X_s and the Jacobian X_J from before.

There are several "Factorization Solutions":

For the Jacobian version of the problem:

$$\begin{aligned} \dot{X} &= \left[X, \Pi_{N_-} X\right] \\ X(0) &= X_0; \end{aligned}$$

if we write

$$\begin{aligned} e^{tX_0} &= n_-(t)\, b_+(t) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ * & \ddots & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix}, \end{aligned}$$

then the solution is

$$X(t) = n_-(t)^{-1} X_0 n_-(t)\,.$$

The flow exists for all time $t \in \mathbb{R}$, as long as $a_i > 0$ for all i. If not all of the a_i are > 0, then the factorization may fail. But you can still do an inverse scattering solution (Kodama, McLaughlin, Ye, 1996).

The symmetric version:

$$\begin{aligned} \dot{X} &= [X, \text{Skew}\,(X)] \\ X(0) &= X_0; \end{aligned}$$

if we write

$$\begin{aligned} e^{tX_0} &= k(t)\, b_+(t) \\ &= (SO(n)\ \text{ matrix}) \begin{pmatrix} * & * & * \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix}, \end{aligned}$$

then the solution is

$$X(t) = k(t)^{-1} X_0 k(t)\,.$$

The flow exists for all time, and factorization always works, for *any* $\alpha_i \in \mathbb{R}$ (Symes, 1979).

Work that has been done on Toda lattice:

Finite nonperiodic Toda lattice

Finite periodic Toda lattice (where you add force between first and last particle)

(semi-)infinite Toda lattices

full-symmetric real X_s: Deift, Li, Nanda, Tomei

tridiagonal, real X_s: Flaschka, Tomei, Kodama, Symes, McLaughlin, Moser complex: full Hessenberg Jacobi form $X_J = \begin{pmatrix} * & 1 & 0 \\ * & \ddots & 1 \\ * & * & * \end{pmatrix}$: Ercolani, Flaschka, Shipman, Singer

complex tridiagonal Jacobi form: Ercolani, Flaschka, Haine
real tridiagonal Jacobi form: Kodama, Ye, Flaschke, Gekhtman
Nonabelian Toda lattices: Bloch, Gekhtman, Koelling
complex semisimple Lie algebras: Gekhtman, Shapiro

2. Geometry of the Toda flows

2.1. Example where factorization fails.

Let

$$\begin{aligned} X_0 &= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\ e^{tX_0} &= \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} \overset{t\neq -1}{=} \begin{pmatrix} 1 & 0 \\ \frac{-t}{1+t} & 1 \end{pmatrix} \begin{pmatrix} 1+t & t \\ 0 & \frac{1}{1+t} \end{pmatrix} \\ &= n_-(t)\, b_+(t). \end{aligned}$$

Then

$$X(t) = n_-^{-1}(t)\, X_0 n_-(t) = \begin{pmatrix} \frac{1}{1+t} & 1 \\ \frac{-1}{(1+t)^2} & \frac{-1}{1+t} \end{pmatrix},\ t \neq -1$$

Then

$$e^{-X_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ t \neq -1$$

If $t \neq -1$, (let $B =$ upper triangular matrices)

$$\begin{aligned} X(t) &\mapsto n_-(t) \operatorname{mod} B \text{ in } SL_2/B \text{ (flag mfld)} \\ &= n_-(t)\, b_+(t) \operatorname{mod} B \\ &= e^{tX_0} \operatorname{mod} B \end{aligned}$$

The latter is defined for all $t \in \mathbb{R}$.

In the flag manifold, when $t = -1$, we have

$$e^{-X_0} \operatorname{mod} B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \operatorname{mod} B$$

Note that

$$\begin{aligned} SL_2/B &= N_- B/B \cup N_- \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B/B \\ &= \text{big cell} \cup \text{pt} \\ &= \text{circle} \end{aligned}$$

So we have completed the flow.

2.2. **Generalizations and geometry.** Jacobi complex version: Let $\mathcal{J}$ be the set of matrices

$$X = \begin{pmatrix} b_1 & 1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & a_{n-1} & b_n \end{pmatrix}$$
$$\text{such that } \sum b_i = 0, a_i, b_i \in \mathbb{C}$$

The (complex) dimension is $2n-2$. The Hamiltonians are

$$H_k = \frac{1}{k+1}\operatorname{tr}\left(X^{k+1}\right)$$

for $k = 1, ..., n-1$. From these you get flows $\dot{X}(t_k) = \left[X(t_k), \Pi_-\left(X^k(t_k)\right)\right]$, $t_k \in \mathbb{C}$. These $n-1$ Hamiltonians cut out an isospectral subset $\mathcal{T}_\Lambda$. If $a_j = 0$, this is preserved under the flow.

Let $e^{t_k X^k} = n(t_k)\, b(t_k) =$ (lower triangular, 1's on diag) (upper triangular $\in B$) . Then the solution is $X(t_k) = n^{-1}(t_k)\, X_o n(t_k)$.

We use the flows to study geometry of $\mathcal{T}_\Lambda$.

The fixed points of the flows are where all $a_j = 0$.

Take $X \in \mathcal{T}_\Lambda$, and say the characteristic poly is

$$\lambda^n - s_2\lambda^{n-2} - s_3\lambda^{n-3} - ... - s_n.$$

The companion matrix is the rational canonical form

$$C_\Lambda = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ s_n & ... & s_3 & s_2 & 0 \end{pmatrix}.$$

Theorem 2.1. *(Kostant) There exists a unique* $L = \begin{pmatrix} 1 & 0 & 0 \\ * & \ddots & 0 \\ * & * & 1 \end{pmatrix}$ *such that*

$$X = LC_\Lambda L^{-1}.$$

The 1-dimensional eigenspace of each λ is spanned by

$$v_\lambda = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix}.$$

Let J_Λ be the Jordan canonical form in $\mathcal{T}_\Lambda$ (with fixed ordering of eigenvalues). Let W be the matrix of generalized eigenvectors so that

$$C_\Lambda = WJ_\Lambda W^{-1}.$$

Take $X_0 \in \mathcal{T}_\Lambda$. Then we write

$$\begin{aligned} X_0 &= LC_\Lambda L^{-1} \\ &= LWJ_\Lambda W^{-1}L^{-1} \end{aligned}$$

We will map X_0 to the flag manifold $SL_n(\mathbb{C})$ mod B. The map is

$$X_0 \mapsto W^{-1}L^{-1}\mathrm{mod}B.$$

Consider

$$\begin{aligned} X(t_k) &= n^{-1}(t_k)\, X_o n(t_k) \\ &= n^{-1}LWJ_\Lambda W^{-1}L^{-1}n, \end{aligned}$$

so $L^{-1}n$ is the unique "L^{-1}" from the Kostant Theorem. Thus, it maps to

$$W^{-1}L^{-1}n(t_k)\,\mathrm{mod}B = W^{-1}L^{-1}e^{t_k X_0^k}\mathrm{mod}B$$

The right hand side is a complete flow (completion of the flow $X(t_k)$).We see

$$\begin{aligned} W^{-1}L^{-1}n(t_k)\,\mathrm{mod}B &= W^{-1}L^{-1}e^{t_k\left(LWJ_\Lambda W^{-1}L^{-1}\right)^k} \\ &= e^{t_k(J_\Lambda)^k}W^{-1}L^{-1}\mathrm{mod}B \end{aligned}$$

This the k^{th} flow in $SL_n(\mathbb{C})$ mod B. So the flow acts on the initial condition. These flows generate an $n-1$-dimensional group action. If the eigenvalues are distinct, this gives the action of the diagonal subgroup on the flag manifold. This group A_Λ is the stabilizer of J_Λ in $SL_n(\mathbb{C})$ under conjugation. If the eigenvalues are not distinct, it is more complicated than the diagonal matrices.

$$A_\Lambda = \text{blocks of} \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & \alpha_2 \\ & & & \alpha_1 \end{pmatrix}.$$

The moment map will map into a polytope in the weight lattice corresponding to the group.

3. The moment map

Let G be a complex semisimple Lie group. Let H be a Cartan subgroup (maximal torus — maximal abelian diagonalizable) — eg diagonal subgroup in $SL(n,\mathbb{C})$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras. Let $\mathfrak{h}_{\mathbb{R}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h},\mathbb{R})$. Let $\sigma : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ (G). Note that $\mathfrak{h}$ acts on V with simultaneous eigenspaces. So

$$V = \bigoplus_{\alpha \in \mathcal{A}} V_\alpha,$$

where $\mathcal{A}$ is the set of weights of V. Note that if $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$,

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} : \sum h_i = 0 \right\}$$

For every $H \in \mathfrak{h}$, if v_α is an eigenvector of $\mathfrak{h}$ then

$$H(v) = \alpha(H)\, v_\alpha$$

So $\alpha \in \mathfrak{h}_{\mathbb{R}}^*$, ie $\alpha : \mathfrak{h} \to \mathbb{R}$ is called the *weight*, and v_α is called a *weight vector*. And V_α is the span of eigenvectors corresponding to weight α. For example, if $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Then

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix} : \sum h_i = 0 \right\}.$$

Let if $v_\alpha = e_1$, so $\alpha_1(h) = h_1$, so $V_{\alpha_1} = \operatorname{span}\{e_1\}$.

Let V be an irreducible representation of $\mathfrak{g}$. Then consider the projectivization $\mathbb{P}(V)$. Then $\mathfrak{g}$ acts on $\mathbb{P}(V)$. There is a highest weight vector $v_{\overline{\alpha}}$. Then you obtain an orbit

$$\begin{aligned} G[v_{\overline{\alpha}}] &\cong G/P \\ &\subset \mathbb{P}(V). \end{aligned}$$

Consider the adjoint representation of $\mathfrak{sl}(3, \mathbb{C})$ on $V = \mathfrak{sl}(3, \mathbb{C})$. Then $\mathbb{P}(\mathfrak{sl}(3, \mathbb{C})) = \mathbb{CP}^7$. Then choose a highest weight vector:

$$\begin{aligned} v_{\overline{\alpha}} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = e_1 \otimes e_3^* \\ &\leftrightarrow e_1 \otimes (e_1 \wedge e_2). \end{aligned}$$

Then $g \in SL(3, \mathbb{C})$ acts on this weight vector by

$$\begin{aligned} g(e_1 \otimes (e_1 \wedge e_2)) &= ge_1 \otimes (ge_1 \wedge ge_2). \\ &= (\text{first column}) \\ &\quad \otimes (\text{span of first two columns}) \end{aligned}$$

So the orbit is a flag manifold, and the stabilizer is

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

Next we want to use the moment map to understand the Toda flows. The moment map is a map

$$\mu : G/P \to \mathfrak{h}_{\mathbb{R}}^*.$$

For $X \in G[v_{\overline{\alpha}}]$, we write it as a linear combination of unit weight vectors:

$$X = \sum_{\alpha \in \mathcal{A}} \Pi_\alpha(X) v_\alpha$$

Then we let

$$\mu([X]) = \frac{\sum_{\alpha \in \mathcal{A}} |\Pi_\alpha(X)|^2 \alpha}{\sum_{\alpha \in \mathcal{A}} |\Pi_\alpha(X)|^2}$$

Then the image $\mu(G/P)$ is the weight polytope of V. Then H acts on G/P by conjugation. The fixed points of the H-action are the vertices of weight polytope. For our example, they are the six points

$$e_i \otimes (e_i \wedge e_j),$$

and the weight polytope is a hexagon.

Theorem 3.1. *(Convexity theorem - Atiyah) Given $[X] \in G[v_{\overline{\alpha}}]$. Then the closure of the torus orbits $\overline{H \cdot [X]}$ = orbit $H \cdot [X]$ with lower-dimensional orbits of H in the boundary.*

Department of Mathematics, University of Texas at Arlington, Arlington, Texas, USA
E-mail address: bshipman@uta.edu