Index theory and quantum field theory: the chiral anomaly

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TCU, Fort Worth, Texas
February 22, 2016

Multi-Particle Quantum Theory (Non-Relativistic)

- $\mathcal{H}$: one-particle Hilbert space (space of wave functions);
- $\mathrm{CAR}(\mathcal{H})$: C^*-algebra generated by symbols $a(v)$ and relations
  - $\{a(v), a(w)\} = 0$
  - $\{a(v), a^*(w)\} = \langle v, w\rangle$
  - $v \mapsto a^*(v)$ linear
- k fermionic particles described by vectors in $\mathcal{F}_k := \Lambda^k\mathcal{H}$;
- Fock space: $\mathcal{F} = \bigoplus_{k=0}^{\infty} \mathcal{F}_k$;
- Vacuum vector: $\Omega := 1 \in \Lambda^0\mathcal{H} = \mathbb{C}$.

Representation of a and a^* as operators on $\mathcal{F}$ given by

$$a^*(v)(w_1 \wedge \ldots \wedge w_k) = v \wedge w_1 \wedge \ldots \wedge w_k,$$
$$a(v)(w_1 \wedge \ldots \wedge w_k) = \iota_v(w_1 \wedge \ldots \wedge w_k).$$

Relativistic Quantum Field Theory

- $P : \mathcal{H} \to \mathcal{H}$ projection onto “states of positive energy”;
- orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_+ = P\mathcal{H}$;
- (p, q)-particle subspaces: $\wedge^p \mathcal{H}_+ \otimes \wedge^q \mathcal{H}_-^*$ in $\mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-^*)$;
- number operator N has eigenvalue $p + q$;
- charge operator Q has eigenvalue $p - q$.

P induces representation of $\mathrm{CAR}(\mathcal{H})$ on $\mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-^*)$ by

$$a(v) = b(Pv) + b^+(\overline{(1 - P)v})$$

Here b^+ and b are creation and annihilation operators on $\mathcal{F}(\mathcal{H}_-^*)$ and $\mathcal{F}(\mathcal{H}_+)$, resp.

Geometric Setup

- X globally hyperbolic Lorentzian spin manifold, even-dimensional, spatially compact;
- $S = S_L \oplus S_R \to X$ spinor bundle;
- $E \to X$ Hermitian vector bundle with connection ∇^E;
- $D : C^\infty(X, S_L \otimes E) \to C^\infty(X, S_R \otimes E)$ Dirac operator;
- advanced/retarded Green's operator $G_\pm : C_0^\infty(X, S_R \otimes E) \to C^\infty(X, S_L \otimes E)$
- one-particle space Hilbert space $\mathcal{H} =$ completion of $\{u \in C^\infty(X, S_L^* \otimes E^*) \mid D^* u = 0\}$;
- well-posedness of Cauchy problem: $\mathcal{H} \xrightarrow{\cong} L^2(\Sigma, S_L^* \otimes E^*)$ for any smooth spacelike Cauchy hypersurface Σ.

The Dirac Quantum Field

Define

$$\begin{aligned} \Psi : C_0^\infty(X; S_R^*X \otimes E^*) &\to \mathrm{CAR}(\mathcal{H}), \\ u &\mapsto a^*(Gu); \\ \overline{\Psi} : C_0^\infty(X; S_L X \otimes E) &\to \mathrm{CAR}(\mathcal{H}), \\ v &\mapsto a(G\overline{v}). \end{aligned}$$

where $G = G_+ - G_-$.
Then Ψ and $\overline{\Psi}$ are $\mathbb{C}$-linear and

$$\{\Psi(u_1), \Psi(u_2)\} = 0, \quad \{\overline{\Psi}(v_1), \overline{\Psi}(v_2)\} = 0;$$

$$\{\overline{\Psi}(v), \Psi(u)\} = -i \int_X \langle v, Gu \rangle \, \mathrm{dV};$$

$$\Psi(u)^* = \overline{\Psi}(\overline{u});$$

$$D\Psi = 0, \quad D^*\overline{\Psi} = 0.$$

States Associated to a Cauchy Hypersurface

- $\Sigma \subset X$ smooth spacelike Cauchy hypersurface;
- spatial Dirac operator D_Σ;
- spectral projector $P_\Sigma = \chi_{[0,\infty)}(D_\Sigma)$;
- $\mathcal{H}_\Sigma = L^2(\Sigma, S_L^* \otimes E^*)$
- wave propagator $U_{\Sigma',\Sigma} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$.

Get induced representation of $\mathrm{CAR}(\mathcal{H})$ on

$$\mathcal{F}(\mathcal{H}) \cong \mathcal{F}(\mathcal{H}_\Sigma) = \mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-^*)$$

where $\mathcal{H}_+ = P_\Sigma \mathcal{H}_\Sigma$.
Associated state:

$$\omega_\Sigma : \mathrm{CAR}(\mathcal{H}) \to \mathbb{C}, \quad x \mapsto \langle \Omega, \pi_\Sigma(x)\, \Omega \rangle$$

Example: two-point function

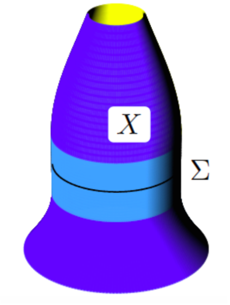
$$\omega_\Sigma^{(2)}(v,u) := \omega_\Sigma(\overline{\Psi}(v)\Psi(u)) = \langle \Omega, \pi_\Sigma(\overline{\Psi}(v)\Psi(u))\, \Omega \rangle$$

distributional bi-solution of Dirac equation

States Associated to aCauchy Hypersurface

$$\Sigma \rightsquigarrow D_\Sigma \rightsquigarrow P_\Sigma \rightsquigarrow \pi_\Sigma \rightsquigarrow \omega_\Sigma$$

Two-point function is Hadamard (has specified singular part) if X has product structure near Σ.



Quantized Dirac Current

Want to quantize classical Dirac current

$$J(X) = \langle \psi, X \cdot \psi \rangle$$

Fix a state ω and try

$$J^{\omega}_{\mu}(x) = \omega(\overline{\Psi}^{\dot{A}}(x)(\gamma_{\mu})^{B}_{\dot{A}}\Psi_{B}(x))$$

Problem: singularities of two-point function. Need regularization procedure (renormalization).
But: relative current

$$J^{\omega_1,\omega_2}_{\mu}(x) = \lim_{y \to x}\left(\omega_1(\overline{\Psi}^{\dot{A}}(x)(\gamma_{\mu})^{B}_{\dot{A}}\Psi_{B}(y)) - \omega_2(\overline{\Psi}^{\dot{A}}(x)(\gamma_{\mu})^{B}_{\dot{A}}\Psi_{B}(y))\right)$$

does exist and is smooth if ω_i are Hadamard!

Charge Creation and Index

Theorem

For any two Cauchy hypersurfaces with product structure near them, the relative current J^{ω_1,ω_2} is conserved (divergence free) and its integral over any Cauchy surface equals

$$\mathrm{ind}((U_{\Sigma',\Sigma})_{++}) = -\mathrm{ind}(D_{\mathrm{APS}}).$$

Hence

$$Q_L = -\int_M \widehat{\mathrm{A}} \wedge \mathrm{ch}(\nabla^E) + \frac{h(D_{\Sigma_1}) - h(D_{\Sigma_2}) + \eta(D_{\Sigma_1}) - \eta(D_{\Sigma_2})}{2}.$$

Similarly

$$Q_R = \int_M \widehat{\mathrm{A}} \wedge \mathrm{ch}(\nabla^E) - \frac{h(D_{\Sigma_1}) - h(D_{\Sigma_2}) + \eta(D_{\Sigma_1}) - \eta(D_{\Sigma_2})}{2}.$$

Total charge $Q = Q_R + Q_L$ is zero.
Chiral charge $Q_{\mathrm{chiral}} = Q_R - Q_L$ charge is not!

Examples

Example 1: $X = \mathbb{R} \times S^3$ with metric $-dt^2 + g_t$ where g_t is a suitable family of *Berger metrics*.
Nontrivial chiral anomaly.

Example 2: (Bianchi-type I spacetimes) $X = \mathbb{R} \times T^3$ with metric $-dt^2 + g_t$ where g_t is any family of flat metrics.
Trivial chiral anomaly.

Example 3: (Bianchi-type II spacetimes) $X = \mathbb{R} \times He(3)$ with metric $-dt^2 + g_t$ where g_t is a suitable family of left-invariant metrics.
Nontrivial chiral anomaly.

References:

C. Bär and A. Strohmaier: *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*
`arXiv:1506.00959`

C. Bär and A. Strohmaier: *A rigorous geometric derivation of the chiral anomaly in curved backgrounds*
`arXiv:1508.05345`

Thank you for your attention!